Philipps Universität Marburg

Geometric structures and special spinor fields

Dissertation

zur Erlangung des Doktorgrades der Naturwissenschaften
(Dr. rer. nat.)

am Fachbereich der Mathematik und Informatik

Philipps-Universität Marburg (Hochschulkennziffer 1180)

von

Dipl. Math. Jos Höll

geboren am 13.05.1984 in Herrenberg

Erstgutachter: Prof. Dr. habil Ilka Agricola (Universität Marburg)
Zweitgutachter: Prof. Dr. Stefan Ivanov (University of Sofia)

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Deutsche Zusammenfassung

Gegen Ende der 1920er Jahre legte Paul Dirac den Grundstein für die Entwicklung von Spinoren und dem natürlicher Weise auf ihnen operierenden Differentialoperator, dem Dirac-Operator ([Di28a, Di28b]). Seit diesem Zeitpunkt spielen Spinoren, die spezielle Differentialgleichungen erfüllen eine große Rolle in Physik und Mathematik. Es stellte sich heraus, dass spezielle Spinoren nur auf bestimmten Typen von Mannigfaltigkeiten existieren können. Es wurden Korrespondenzen zwischen Differenzialgleichungen für Spinoren und Typen von Mannigfaltigkeiten entdeckt. Einige wichtige Zusammenhänge sind in der folgenden Liste aufgeführt.

- Parallele Spinoren existieren nur auf Ricci-flachen Räumen.
- Der Index des Dirac-Operators ist gleich dem (rein topologischen) $\hat{A}$ Geschlecht [AS62].
- Parallele Spinoren erfordern Holonomie $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$, G_2 und Spin(7) [Wa89].
- Korrespondenzen zwischen Killing-Spinoren und geometrischen Strukturen [FK89, FK90, Gr90].

Wir werden die im letzten Punkt genannten Killing-Spinoren verallgemeinern, wie es beispielsweise schon in [FK01, BGM05] getan wurde, um damit neue Zusammenhänge mit geometrischen Strukturen herzustellen. So korrespondieren zum Beispiel in Dimension 6 die generalisierten Killing-Spinoren, die für den Levi-Civita Zusammenhang ∇ mit einem symmetrischen Endomorphismus S durch die Gleichung

$$\nabla_X \phi = S(X) \cdot \phi$$

beschrieben werden, zu halb-flachen SU(3)-Strukturen. Ebenso ergeben sich Korrespondenzen zum Kern des Dirac-Operators, wie sie in der Physik von großem Interesse sind; in [CCD03] wurden beispielsweise die Restriktionen für eben diese Korrespondenz bereits betrachtet.

Im ersten Kapitel werden die geometrischen Strukturen eingeführt, die in dieser Arbeit behandelt werden und die oben genannten Korrespondenzen zu Spinoren untersucht. Im zweiten Kapitel wenden wir uns der Hyperflächentheorie zu. Ein Spinor ϕ auf einer Mannigfaltigkeit $\bar{M}$ mit Hyperfläche $M \subset \bar{M}$ erfüllt für die zugehörigen Levi-Civita Zusammenhänge ∇ und $\bar{\nabla}$ die Gleichung

$$\bar{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} W(X) \cdot \phi, \tag{1}$$

wobei W die Weingarten-Abbildung ist (z.B. [BGM05]). Dies gibt uns die Möglichkeit, Zusammenhänge von Spinoren und durch den ersten Teil der Arbeit damit eben auch zwischen geometrischen Strukturen auf M und $\bar{M}$ herzustellen.
Die meisten geometrischen Strukturen tragen einen charakteristischen Zusammenhang ∇^c mit Torsion T, der metrisch ist, die Struktur erhält und außerdem die gleichen Geodäten wie der Levi-Civita Zusammenhang ∇ besitzt. Wir werden diese Zusammnhänge nutzen, um Korrespondenzen von geometrischen Strukturen auf Untermannigfaltigkeiten zu schaffen, die nicht durch Spinoren gegeben sind.
Generalisierte Killing-Spinoren, wie sie oben beschrieben wurden, und Killing-Spinoren mit Torsion, definiert durch die Gleichung

$$\nabla^s_X \phi = \alpha X \cdot \phi \tag{2}$$

für den Zusammenhang $\nabla^s = \nabla + 2sT$, wobei ∇ der Levi-Civita Zusammenhang und T die charakteristische Torsion ist, sind von immer größerer Bedeutung (siehe z.B. [ABBK13, FK01, BGM05]). Wir werden charakteristische Zusammenhänge nutzen, um solche Spinoren auf M und $\bar{M}$ zu untersuchen.

In Abschnitt 1 des ersten Kapitels werden metrische fast-Kontakt-Strukturen eingeführt. Hier wird an die Klassifikation solcher Strukturen erinnert und außerdem ein nützliches Kriterium

für die Existenz eines charakteristischen Zusammenhangs gegeben. Wir nutzen dieses Kriterium, um zu zeigen, dass in der Klasse $\mathcal{C}_{13467}$ der Klassifikation von Chinea und Gonzalez ([CG90]) ein charakteristischer Zusammenhang existiert, nicht aber in $\mathcal{C}_2$, $\mathcal{C}_5$, $\mathcal{C}_9$, $\mathcal{C}_{10}$, $\mathcal{C}_{11}$ oder $\mathcal{C}_{12}$ (Theorem 1.3).
Als nächstes werden in Abschnitt 2 fast-hermitsche Strukturen definiert und ebenfalls deren Klassifikation sowie das bekannte Kriterium ([FI02]) zur Existenz charakteristischer Zusammenhänge wiederholt.
Im speziellen Fall der Dimension 6 werden in Abschnitt 3 SU(3)-Strukturen eingeführt. Da die Gruppe $SU(3) \subset Spin(6)$ der Stabilisator eines Spinors ist, können hier Korrespondenzen zwischen dem definierenden Spinor und der resultierenden Struktur geschaffen werden. So wird hier die Klassifikation der SU(3)-Strukturen in spinorielle Gleichungen übersetzt. Es werden Größen wie der Nijenhuis-Tensor und der Dirac-Operator betrachtet. So ist beispielsweise das Kodifferential der Kählerform durch den Dirac-Operator D und die klassifizierende 1-Form η gegeben mittels

$$\delta\omega(X) = 2[(D\phi, X \cdot j \cdot \phi) - \eta(X)].$$

Zudem wird eine sehr interessante Klasse von SU(3)-Zusammenhängen betrachtet sowie eine spinorielle Bedingung für die Existenz eines charakteristischen Zusammenhangs entwickelt.
Ähnlich wie im vorherigen Abschnitt wird in Abschnitt 4 eine Korrespondenz zwischen Spinoren und G_2-Strukturen in Dimension 7 geschaffen. Beispielsweise ist ein Spinor der Länge 1 genau dann im Kern des Dirac-Operators, wenn die zugehörige G_2-Struktur aus der Klasse $\mathcal{W}_{23}$ ist. Wie im vorherigen Abschnitt in Dimension 6, werden auch hier an die G_2-Geometrie angepasste Zusammenhänge näher erläutert.
In Abschnitt 5 werden kurz Spin(7)-Strukturen in Dimension 8 eingeführt. Da ein Spinor in dieser Dimension jedoch nicht immer die Gruppe Spin(7) als Stabilisator besitzt, kann hier keine Korrespondenz wie in den obigen Fällen gegeben werden.

Im zweiten Kapitel liegt der Fokus auf der Hyperflächentheorie. Trägt eine Mannigfaltigkeit $\bar{M}$ mit Hyperfläche M eine bestimmte G-Struktur so lässt sich diese in eine andere geometrische Struktur auf M überführen. Mittels der Beziehung aus Gleichung (1) und der jeweilig definierenden Spinoren wird im Fall $\dim M = 6$ und $\dim \bar{M} = 7$ in Abschnitt 1 eine Korrespondenz zwischen G_2-Strukturen und deren SU(3)-Hyperflächen ausgearbeitet. Es werden außerdem verallgemeinerte Killing-Spinoren mit Torsion (eine Verallgemeinerung der Gleichung (2)) eingeführt und deren Korrespondenzen auf M und $\bar{M}$ bestimmt. Wir nutzen die Kegelkonstruktion, um aus einer SU(3)-Mannigfaltigkeit eine G_2-Mannigfaltigkeit eines bestimmten Typs zu konstruieren. Außerdem werden Beziehungen zwischen generalisierten Killing-Spinoren mit Torsion auf M und $\bar{M}$ geschaffen.
Aus dem oben im Fall der Spin(7)-Struktur erwähnten Grund lassen sich in anderen Dimensionen als $\dim M = 6$ und $\dim \bar{M} = 7$ mittels Spinoren keine Beziehungen für die von uns betrachteten geometrischen Strukturen auf Hperflächen herstellen. Daher werden in Abschnitt 2 Zusammenhänge mit Torsion betrachtet. Diese sind im Folgenden ein wichtiges Werkzeug für die Untersuchung von Korrespondenzen zwischen geometrischen Strukturen auf M und dem Kegel $\bar{M}$, selbst wenn diese Strukturen nicht durch Spinoren gegeben sind. In [Bä93] wird diese Konstruktion benutzt, um eine Beziehung zwischen Riemannschen Killing-Spinoren und der geometrischen Struktur zu schaffen. Wir benutzen diese Zusammenhänge außerdem, um Killing-Spinoren mit Torsion auf Hyperflächen zu betrachten.
So wird in Abschnitt 3 auf dem Kegel einer fast-Kontakt-Struktur eine fast-hermitsche Struktur konstruiert und die jeweiligen Klassifikationen mit einander in Beziehung gesetzt. Wir zeigen beispielsweise, dass eine α-Sasaki Struktur zu einer lokal-konform-Kähler Struktur in Beziehung steht oder dass die beiden Nijenhuis-Tensoren die Gleichen sind. Wir benutzen die in Abschnitt 2 geschaffenen Korrespondenzen zwischen Spinoren, um zu zeigen, dass ein Killing-Spinor mit Torsion auf (M, g) einem Spinor ϕ auf dem Kegel $(\bar{M}, \bar{g}) = (M \times \mathbb{R}, a^2 r^2 g + dr^2)$, der die Bedin-

gung

$$\bar{\nabla}^c_X \phi + \frac{1}{2r}(X \lrcorner (\partial_r \lrcorner \omega) \wedge \omega)\phi = 0$$

erfüllt, entspricht. Hier ist ω die Kähler-Form und $\bar{\nabla}^c$ der charakteristische Zusammenhang auf $\bar{M}$.
In Abschnitt 3.4 lässt sich anhand der Tatsache, dass die aus den 3 fast-Kontakt-Strukturen konstruierten Zusammenhänge nicht die selben sind, nicht wie in den übrigen Abschnitten verfahren und wir belassen es bei einer kurzen Betrachtung der Situation.
Im Falle einer G_2-Struktur auf einer 7 dimensionalen Mannigfaltigkeit M trägt der Kegel eine Spin(7)-Struktur. Dieser Fall sowie die Korrespondenz der Klassifikationen und der generalisierten Killing-Spinoren mit Torsion ist in Abschnitt 4 ausgearbeitet. Hier wird beispielsweise bewiesen, dass eine Spin(7)-Struktur der Klasse $\mathcal{U}_1$ (bzw. $\mathcal{U}_2$) auf dem Kegel eine G_2 Struktur auf M induziert, die niemals aus der Klasse $\mathcal{W}_{34}$ (bzw. $\mathcal{W}_{13}$) sein kann.
Einen Teil dieser Ergebnisse (im Wesentlichen sind dies die Ergebnisse des Abschnittes 1 aus Kapitel I sowie die Abschnitte 2, 3 und 4 aus Kapitel II) haben wir bereits in [AH13] publiziert.

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Contents

Introduction

The idea of spinors and the Dirac operator as the natural differential operator acting on them was introduced by Paul Dirac in the late 1920's [Di28a, Di28b]. Since then spinors played an important role in mathematics and physics. Spinors fulfilling special differential equations were of great interest already to Paul Dirac. The observation that special spinors require a certain type of manifold to live on was developed. The earliest example of this fact is that the existence of a parallel spinor field on a Riemannian manifold requires the manifold to be Ricci-flat. This brings out the fact, that the existence of a solution to a differential equation imposes strong conditions in the geometry. The correspondences of manifolds carrying different geometric structures and the appendant spinors fulfilling interesting equations is one main point of this thesis.
The most popular and important correspondence of special spinors is the Atiyah-Singer index theorem (see [AS62]), which states that the purely topological $\hat{A}$ genus of a compact Riemannian spin manifold is equal to the index of the Dirac operator (in [AS62] for simplicity $\dim \equiv 0 \mod 8$ is assumed).
Another milestone was the list of Berger (see [Be55, Si62]). He determined the Ricci-flat Riemannian holonomy groups, which thus are candidates for manifolds with parallel spinors. The theorem of Wang in 1989 (see [Wa89]) shows us, that these groups indeed appear. He proved that a complete simply connected irreducible non-flat Riemannian spin manifold carries a parallel spinor if and only if its Riemannian holonomy is

- $\mathrm{Sp}(m)$ in dimension $4m$,
- $\mathrm{SU}(m)$ in dimension $2m$,
- G_2 in dimension 7 or
- $\mathrm{Spin}(7)$ in dimension 8.

We define a G structure to be a reduction of the frame bundle of (M, g) to a G bundle. M is then a so called G manifold.
Fix a group $G \subset \mathrm{SO}(n)$. Then the classification of G structures is based on the following concept of intrinsic torsion. Given a G structure, the Levi-Civita connection one form has values in the corresponding Lie algebra $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of the Lie algebra $\mathfrak{g}$ of G in $\mathfrak{so}(n)$. The $\mathfrak{m}$ part of this one form is the so called intrinsic torsion and the space $\mathfrak{m}$ splits in irreducible representations $\mathfrak{m} = \mathfrak{m}_1 \oplus .. \oplus \mathfrak{m}_k$ under G. If only the $\mathfrak{m}_l$ part is non-zero, the structure is said to be of class $\mathfrak{m}_l$ for some l. Popular classes have their own names. For example in dimension 6 and for $G = \mathrm{U}(3)$ we have nearly Kähler structures, almost Kähler structures and many others. Important G_2 structures in dimension 7 are for example nearly parallel or cocalibrated. If the Levi-Civita one form takes values in $\mathfrak{g}$, a structure is said to be integrable. We are interested in the non-integrable case. Then the (non-zero) intrinsic torsion can be used not only for classification but to construct a connection adapted to the G structure. This connection is an important tool in the investigation of G structures as described later. With the list above this gives us a correspondence of integrable $\mathrm{SU}(n)$ (respectively $\mathrm{Sp}(n)$, $\mathrm{Spin}(7)$ or G_2) structures and parallel spinors.

Other correspondences between spinors and geometric structures came up in terms of Killing spinors. On a Riemannian spin manifold, a spinor ϕ is said to be a Riemannian Killing spinor if it satisfies

$$\nabla_X \phi = \alpha X \cdot \phi$$

for the Levi-Civita connection ∇, the Clifford multiplication $\cdot$ and some constant $\alpha \in \mathbb{C}$. Killing spinors are geometrically interesting as they realize the limiting case of the lower bound for the eigenvalue of the Dirac operator (see [Fr80]). Again, the existence of a spinor satisfying such an equation strongly restricts the geometry. There exists a real ($\alpha \in \mathbb{R}$) Killing spinor on a n-dimensional manifold if it carries

- an Einstein Sasaki structure in dimension $n = 5$ ([FK89]),

- a nearly Kähler structure in dimension $n = 6$ ([Gr90]),
- a nearly parallel G_2 structure in dimension $n = 7$ ([FK90]).

In dimension 8, real Killing spinors exist only on the sphere ([Hi86], [BFGK91] on page 123). As this is a great restriction to the geometry (for example in dimension 6, there are only some examples known, see [Gr90, FG85]), generalizations of the Killing equation become more and more interesting (see [ABBK13, FK01, BGM05] and others), which also has to do with the following fact.
A spinor does not only define a geometric structure if it is a parallel spinor or a Killing spinor. In dimensions 6 and 7, the stabilizer of a spinor is SU(3) respectively G_2 and we are able to translate geometric data and classifications from structures to spinors and vice versa.
In sections 3 and 4 of Chapter I we will introduce the classifications of SU(3) and G_2 structures as they were developed in [CS02] and [FG82] and describe them with spinorial equations. For example we will see, that to any spinor ϕ of length one in dimension 6 (resp. $\bar{\phi}$ in dimension 7) there always exists a one form η and an endomorphism S (resp. an endomorphism $\bar{S}$) such that

$$\nabla_X \phi = \eta(X) j \cdot \phi + S(X) \cdot \phi \qquad \text{(resp. } \nabla_X \bar{\phi} = \bar{S}(X) \cdot \bar{\phi}\text{)}, \tag{3}$$

where $j = e_1 \cdot .. \cdot e_6$ for any local basis e_i. If $\eta = 0$ and S is symmetric (resp. if $\bar{S}$ is symmetric) then ϕ (resp. $\bar{\phi}$) is called a *generalized Killing spinor* (see [FK01] and [BGM05]) and corresponds to a half flat SU(3) structure in dimension 6 and a cocalibrated G_2 structure in dimension 7. If in addition S (resp. $\bar{S}$) is a multiple of the identity this reduces to the Killing equation. Also we see, that there are interesting correspondences to the Dirac operator. An SU(3) structure of type $\chi_{2\bar{2}345}$ with a certain restriction on the χ_{45} part corresponds to a spinor of length one in the kernel of the Dirac operator (see Theorem 3.9).
Such equations involving the Dirac operator are also interesting in physics. The restrictions mentioned already came up in the work of Cardoso and others (see [CCD03]).

In hypersurface theory, generalized Killing spinors play an important role (see for example [BGM05]), since on a hypersurface M in $\bar{M}$ with Levi-Civita connections ∇ and $\bar{\nabla}$ a spinor ϕ satisfies

$$\bar{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} W(X) \cdot \phi, \tag{4}$$

where W is the symmetric Weingarten tensor. If $\dim M = 6$ and $\dim(\bar{M}) = 7$, we are able to look at a spinor defining an SU(3) structure on M and a G_2 structure on $\bar{M}$. Using Equation (3), the classification of M and $\bar{M}$ can then be given in terms of each other.

Many investigations of the correspondence between spinors and geometric structures are done in other dimensions then 6 and 7 as well (see for example [Iv04] for dimension 8 or [FI03] for dimension 5) but there is not always an applicable correspondence. So, to compare classifications on M and $\bar{M}$ of structures not given by a spinor we need a different tool.
Correspondences of Killing spinors on a manifold M and parallel spinors its cone $\bar{M}$ were first recognized by Bryant ([Br87]) in some examples. In [Bä93] Bär translated the existence of Killing spinors on a manifold M to the classification of parallel spinors by Wang on its cone $\bar{M}$. We will generalize this construction using connections with torsion to compare the classification on M of almost contact structures (respectively G_2 structures) and the classification on $\bar{M}$ of almost hermitian structures (respectively Spin(7) structures).
If the projection of the intrinsic torsion to the 3-forms is non-zero, it defines the skew symmetric torsion of a metric connection preserving the geometric structure (see [FI02]), which is typically unique (see [AFH13] for the most general case) and thus is called characteristic connection. This connection not only is metric and preserves the G structure, it also has the same geodesics as the Levi-Civita connection. The characteristic connections on M and $\bar{M}$ can be used for comparison of two geometric structures. The case of almost contact structures on M and almost

hermitian structures on $\bar{M}$ was already discussed in physics, see [HTY12], in a less general setting.

Another generalization of Killing spinors is constructed using the characteristic connection. A spinor ϕ is said to be a Killing spinor with torsion, if it satisfies

$$\nabla_X \phi + s(X \lrcorner T) \cdot \phi = \alpha X \cdot \phi \tag{5}$$

for some $s \in \mathbb{R}$, where T is the characteristic torsion. Killing spinors with torsion became interesting in the last years, since for example for $s = \frac{n-1}{4(n-3)}$ they realize the equality case of the eigenvalue estimation of the Dirac operator with torsion (which in some cases is also known as the cubic Dirac operator or the Dolbeault operator), see [ABBK13]. Also, much more examples can be constructed, since the restriction to the geometry given by the existance of a Killing spinor with torsion is not as strong as the restriction given by a Riemannian Killing spinor. This richness implies, that a classification is not possible. Using Equations (5) and (4) we are able to give correspondences of spinors satisfying generalized Killing equations on M and its cone, or in some cases even correspondences of spinors on a general hypersurface M and the ones on its ambient space.

In the first Chapter we will introduce the geometric structures which will be used in this thesis. We will start with metric almost contact structures in Section 1. We cite the classification of such, given by Chinea and Gonzalez in [CG90] and give a useful criterion of the existence of a characteristic connection in Section 1.1. We use this criterion to see that for an almost contact manifold there exists a characteristic connection if it is of type $\mathcal{C}_{13467}$ but not, if it is of pure type $\mathcal{C}_2$, $\mathcal{C}_5$, $\mathcal{C}_9$, $\mathcal{C}_{10}$, $\mathcal{C}_{11}$ or $\mathcal{C}_{12}$ (Theorem 1.3).
We shortly introduce almost hermitian structures in Section 2. The criterion for the existence of a characteristic connection in this case is already known (see [FI02]).
As mentioned before, special almost hermitian structures in dimension 6 are given as the stabilizer of a spinor. In Section 3 we will describe this correspondence in detail (see Lemma 3.1). To understand this concept we will shortly introduce the corresponding spin linear algebra (Section 3.1) to then give a spinorial description of the intrinsic torsion and the classification of SU(3) structures (Sections 3.2 and 3.3). In addition to the Dirac Operator as mentioned above, we calculate the Nijenhuis tensor in terms of the defining spinor (see Lemma 3.14). We will get useful equations as the following. For the Dirac operator D and the real inner product $(\, , \,)$ of spinors the codifferential of the Kähler form ω in terms of the defining spinor ϕ is given by (see Lemma 3.8)

$$\delta\omega(X) = 2[(D\phi, X \cdot j \cdot \phi) - \eta(X)],$$

where η is the intrinsic one form. In terms of spinors the intrinsic torsion can easily be used to define certain SU(3) connections, which we will introduce in Section 3.4. Here are given some tools to handle the characteristic connection and to show that an SU(3) manifold carries a characteristic connection if it is of type $\chi_{1\bar{1}345}$ with a certain restriction on the 4 and 5 part of the intrinsic torsion (Theorem 3.22).
In Section 4 the same is done for the structure group G_2 in dimension 7. The correspondence of G_2 structures and spinors is given in Lemma 4.1. For the classification in terms of spinors in Section 4.3 we calculate correspondences of the following sort. A spinor ϕ of constant length is in the kernel of the Dirac operator if and only if it defines a G_2 structure of type $\mathcal{W}_{23}$. As in the SU(3) case in Section 4.4 we give a description of the torsion for the characteristic connection in terms of the defining spinor (see Theorem 4.14).
In dimension 8 a spinor does not always have stabilizer Spin(7), so we were not able to give correspondences as in the 6 and 7 dimensional case. Here, there always exists a characteristic connection and so we only shortly introduce Spin(7) structures as they will be used in this thesis.

In Chapter II we concentrate on hypersurface theory and the special case of a cone construction. As demonstrated in Equation (4) a spinor on a manifold $\bar{M}^7$ defining a G_2 structure can be viewed as a spinor on a hypersurface M^6 inducing an SU(3) structure. The classification of both can be expressed in terms of each other as we will see in Theorems 1.4 and 1.5 of Section 1. The special case of a (twisted) cone $\bar{M}$ over an SU(3) manifold is considered in Section 1.1. With this tool we are able to construct G_2 structures of different types, starting with a certain SU(3) manifold. Killing spinors with torsion are the topic of Section 1.2 as they correspond to certain spinors on the cone or a more general ambient space. This correspondence is worked out in Theorem 1.8.

For hypersurfaces of other dimension then 6, interesting G structures are not always given by a spinor, so we need another tool as described above and have to restrict ourselves to the case of a (twisted) cone. In Section 2.1 we introduce the construction of a twisted cone as it was done in a less general case by Bär in [Bä93]. We make extensive use of characteristic connections and connections of the form described in Equation (5) for some T. Starting with a Riemannian manifold M with characteristic connection ∇, in Lemma 2.4 we prove, that a spinor on the cone being parallel for a certain connection with torsion corresponds a ∇-Killing spinor on M. Section 2 provides the tools we will apply in the next two sections to certain dimensions and G structures.

In Section 3 we concentrate on a manifold M with almost contact structure. A twisted cone over such a manifold carries an almost hermitian structure (Theorem 3.2) and with the tools described above, we are able to compare the two classifications (Section 3.1). We see for example that an α-Sasaki structure corresponds to a locally conformally Kähler structure on the cone (Theorem 3.12). In Lemma 3.7 we additionally prove, that the two Nijenhuis tensors are basically the same. We also apply the spinorial correspondences of Section 2.2 to this case to get interesting spinorial equations in terms of the data of the geometric structure. For example we get a one to one correspondence between Killing spinors with torsion on (M, g) and spinors ϕ on the cone $(\bar{M}, \bar{g}) = (M \times \mathbb{R}, a^2 r^2 g + dr^2)$ for some fixed $a > 1$ satisfying

$$\bar{\nabla}^c_X \phi + \frac{1}{2r}(X \lrcorner (\partial_r \lrcorner \omega) \wedge \omega)\phi = 0,$$

where ω is the Kähler form and $\bar{\nabla}^c$ is the characteristic connection on $\bar{M}$. For examples of this case see Section 3.3. We also take a quick look on metric almost contact 3-structures (the more general case of a 3-Sasakian structure) in Section 3.4. But since the connections we construct to each of the three almost contact structures do not coincide, we shall only make a few comments here. However, in dimension 7, 3-Sasakian manifolds carry a cocalibrated G_2 structure, which then has a characteristic connection ([AF10]). This case is discussed in terms of G_2 structures in Section 4, Example 4.18.

We continue with the investigation in dimension 7. We look at G_2 structures and their corresponding Spin(7) structures on the cone to compare the two classifications in Section 4.1. In Theorem 4.13 we show, that a Spin(7) structure of type $\mathcal{U}_1$ on the cone induces a G_2 structure, which is never of type $\mathcal{W}_{34}$ and that a structure of type $\mathcal{U}_2$ leads to a G_2 structure, which cannot be of type $\mathcal{W}_{13}$. Again we calculate correspondences of spinors on a G_2 manifold and spinors on its Spin(7) cone in terms of the geometric data to give interesting examples in Section 4.3. Some of this results (mainly the results from Section 1 of Chapter I and Sections 2, 3 and 4 from Chapter II) we already published in [AH13].

Chapter I

G structures and their characteristic connections

Let (M, g) be an oriented Riemannian manifold with Levi-Civita connection ∇^g witch connection 1-form Z. By definition, a G structure on M is a reduction of the frame bundle of M to some closed subgroup $G \subset \mathrm{SO}(n)$. For the classification of such structures we consider the connection 1-form Z with values in $\mathfrak{so}(n)$. We decompose

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m},$$

where $\mathfrak{g}$ is the Lie algebra of G and consider the corresponding splitting $Z = Z^* + \Gamma$. Then Γ is called *intrinsic torsion* of the G structure. Again we decompose $\mathfrak{m}$ into irreducible representations of G giving the classes we use for classification. In some cases we will look at the connection Z^* with linear connection ∇^n and calculate its connection type given by the decomposition of the space of all metric connections

$$TM \oplus \Lambda^3(TM) \oplus \mathcal{T}$$

where the parts are called *vectorial*, *skew symmetric* and *cyclic traceless*. See Section 3.4 for more details on this decomposition.

If M admits a metric connection ∇^c with skew symmetric torsion T^c preserving the G structure, it will be called a *characteristic connection*. This is a metric connection which is adapted to the structure (rather then ∇^g) but still has the same geodesics then ∇^g. In sections 3 and 4 of this chapter and in Section 1 of Chapter II we will mostly work with the Levi-Civita connection and thus shorten ∇^g to ∇, while in the other sections the characteristic connection is used more frequently and thus often ∇^c will be shortened to ∇.

The following result proves the uniqueness of the characteristic connection in many geometric situations:

Theorem 0.1 ([AFH13, Thm 2.1.]). *Let $G \subsetneq \mathrm{SO}(n)$ be a connected Lie subgroup acting irreducibly on $\mathbb{R}^n$, and assume that G does not act on $\mathbb{R}^n$ by its adjoint representation. Then the characteristic connection of a G structure on a Riemannian manifold (M, g) is, if existent, unique.*

This applies, for example, to almost hermitian structures ($\mathrm{U}(n) \subset \mathrm{SO}(2n)$), G_2 structures in dimension 7 and $\mathrm{Spin}(7)$ structures in dimension 8 (but not to metric almost contact structures). We will now introduce the G structures considered in this thesis.

1 Metric almost contact structures

Let M be a $n = 2k+1$ dimensional manifold. Given a Riemannian metric g, a (1,1)-tensor $\psi : TM \to TM$, a 1-form η with dual vector field ξ of length one, and the $(2,0)$-tensor F defined by $F(v,w) := g(v, \psi(w))$, we call (M, g, ψ, η) a metric almost contact structure if

$$\psi^2 = -id + \eta \otimes \xi \qquad \text{and} \qquad g(\psi v, \psi w) = g(v,w) - \eta(v)\eta(w).$$

In [Bl02, Thm 4.1.D], D. Blair shows that $\psi(\xi) = 0$ and $\eta \circ \psi = 0$. Since

$$g(v,\psi(w)) \;=\; g(\psi(v), \psi^2(w)) + \eta(v)\eta(\psi(w)) \;=\; g(\psi(v), -w + \eta(w)\xi) \;=\; -g(\psi(v), w),$$

for all $v, w \in TM$, F is actually a 2-form. In terms of the Levi-Civita connection ∇^g on M, the Nijenhuis tensor of a metric almost contact structure is defined by

$$\begin{aligned} N(X,Y,Z) \;:=\; & g((\nabla^g_X \psi)(\psi(Y)) - (\nabla^g_Y \psi)(\psi(X)) + (\nabla^g_{\psi(X)} \psi)(Y) - (\nabla^g_{\psi(Y)} \psi)(X), Z) \\ & + \eta(X) g(\nabla^g_Y \xi, Z) - \eta(Y) g(\nabla^g_X \xi, Z). \end{aligned}$$

The classification of metric almost contact structures is relatively involved. For future reference, we recall in the following table the exact definition of the different classes of of n-dimensional metric almost contact manifolds given by Chinea and Gonzalez [CG90].

class	defining relation
$\mathcal{C}_1$	$(\nabla^g_X F)(Y,Z) = 0,\ \nabla^g \eta = 0$
$\mathcal{C}_2$	$dF = \nabla^g \eta = 0$
$\mathcal{C}_3$	$(\nabla^g_X F)(Y,Z) - (\nabla^g_{\psi X} F)(\psi Y, Z) = 0$
$\mathcal{C}_4$	$(\nabla^g_X F)(Y,Z) = -\frac{1}{n-3}[g(\psi X, \psi Y)\delta F(Z) - g(\psi X, \psi Z)\delta F(Y)$ $-F(X,Y)\delta F(\psi Z) + F(X,Z,\delta F(\psi Y)], \quad \delta F(\xi) = 0$
$\mathcal{C}_5$	$(\nabla^g_X F)(Y,Z) = \frac{1}{n-1}[F(X,Z)\eta(Y) - F(X,Y)\eta(Z)]\delta\eta$
$\mathcal{C}_6$	$(\nabla^g_X F)(Y,Z) = \frac{1}{n-1}[g(X,Z)\eta(Y) - g(X,Y)\eta(Z)]\delta F(\xi)$
$\mathcal{C}_7$	$(\nabla^g_X F)(Y,Z) = \eta(Z)(\nabla^g_Y \eta)(\psi X) + \eta(Y)(\nabla^g_{\psi X}\eta)(Z), \quad \delta F = 0$
$\mathcal{C}_8$	$(\nabla^g_X F)(Y,Z) = -\eta(Z)(\nabla^g_Y \eta)(\psi X) + \eta(Y)(\nabla^g_{\psi X}\eta)(Z), \quad \delta\eta = 0$
$\mathcal{C}_9$	$(\nabla^g_X F)(Y,Z) = \eta(Z)(\nabla^g_Y \eta)(\psi X) - \eta(Y)(\nabla^g_{\psi X}\eta)(Z)$
$\mathcal{C}_{10}$	$(\nabla^g_X F)(Y,Z) = -\eta(Z)(\nabla^g_Y \eta)(\psi X) - \eta(Y)(\nabla^g_{\psi X}\eta)(Z)$
$\mathcal{C}_{11}$	$(\nabla^g_X F)(Y,Z) = -\eta(X)(\nabla^g_\xi F)(\psi Y, \psi Z)$
$\mathcal{C}_{12}$	$(\nabla^g_X F)(Y,Z) = \eta(X)\eta(Z)(\nabla^g_\xi \eta)(\psi Y) - \eta(X)\eta(Y)(\nabla^g_\xi \eta)(\psi Z)$

The most important classes are

- $\mathcal{C}_3 \oplus .. \oplus \mathcal{C}_8$, the normal structures characterized by $N = 0$,
- $\mathcal{C}_6 \oplus C_7$, the quasi Sasaki structures: normal structures satisfying $dF = 0$,
- $\mathcal{C}_6$, the α-Sasaki structures: normal structures with $\alpha F = d\eta$ for some constant α,
- Sasaki structures: α-Sasaki structures with $\delta F(\xi) = n - 1$.

Other classifications we will not consider here are formulated in terms of the Nijenhuis tensor or by considering the direct (not the twisted) product $M \times \mathbb{R}$ ([CM92] and [Ou85]). It turns out that the tensor $\alpha(X,Y,Z) := (\nabla^g_X F)(Y,Z)$ will be a useful tool for the investigation of metric almost contact structures. It satisfies the general formula

$$\alpha(X,Y,Z) \;=\; -\alpha(X,Z,Y) \;=\; -\alpha(X,\psi Y,\psi Z) + \eta(Y)\alpha(X,\xi,Z) + \eta(Z)\alpha(X,Y,\xi). \tag{I.1}$$

This implies

$$\alpha(X,Y,\psi Y) \;=\; -\alpha(X,\psi Y,\psi^2 Y) + \eta(Y)\alpha(X,\xi,\psi Y) \;=\; -\alpha(X,Y,\psi Y) + 2\eta(Y)\alpha(X,\xi,\psi Y),$$

so we have

$$\alpha(X,Y,\psi Y) = \eta(Y)\alpha(X,\xi,\psi Y). \tag{I.2}$$

1.1 Almost contact connections

A metric almost contact structure admits a characteristic connection if and only if its Nijenhuis tensor is skew symmetric and ξ is a Killing vector field, and then it is unique [FI02, Thm 8.2]. If it exists, its torsion tensor is given by

$$T \;=\; \eta \wedge d\eta + dF^\psi + N - \eta \wedge (\xi \lrcorner N),$$

where $dF^\psi := dF \circ \psi$. We shall now prove a useful criterion for the existence of a characteristic connection.

Lemma 1.1. *A metric almost contact manifold (M,g,ψ,η) admits a characteristic connection if and only if*

$$(\nabla^g_Y F)(Y,\psi X) + (\nabla^g_{\psi Y} F)(Y,X) \;=\; 0.$$

Proof. There exists a characteristic connection if and only if the Nijenhuis tensor N is skew symmetric and ξ is a Killing vector field. Since we have

$$g(\nabla^g_Y \xi, Z) \;=\; -F(\nabla^g_Y \xi, \psi Z) \;=\; (\nabla^g_Y F)(\xi,\psi Z) \;=\; (\nabla^g_Y \eta)(Z)$$

and $(\nabla^g_X F)(Z,Y) = g((\nabla^g_X \psi)Y, Z)$, the Nijenhuis tensor on M may be written as

$$\begin{aligned} N(X,Y,Z) \;&= \alpha(X,Z,\psi Y) - \alpha(Y,Z,\psi X) + \alpha(\psi X,Z,Y) - \alpha(\psi Y,Z,X) \\ &\quad + \eta(X)\alpha(Y,\xi,\psi Z) - \eta(Y)\alpha(X,\xi,\psi Z). \end{aligned}$$

Thus N is skew symmetric if

$$0 = N(X,Y,Y) = \alpha(X,Y,\psi Y) - \alpha(Y,Y,\psi X) - \alpha(\psi Y,Y,X) + \eta(X)\alpha(Y,\xi,\psi Y) - \eta(Y)\alpha(X,\xi,\psi Y).$$

With equation (I.2), N is skew symmetric if and only if

$$0 \;=\; -\alpha(Y,Y,\psi X) - \alpha(\psi Y,Y,X) + \eta(X)\alpha(Y,\xi,\psi Y). \tag{I.3}$$

ξ is a Killing vector field if $0 = g(\nabla^g_X \xi, Y) + g(\nabla^g_Y \xi, X) = \alpha(X,\xi,\psi Y) + \alpha(Y,\xi,\psi X)$, and this is satisfied if and only if $\alpha(Y,\xi,\psi Y) = 0$. Together with condition (I.3) we obtain the condition

$$0 \;=\; \alpha(Y,Y,\psi X) + \alpha(\psi Y,Y,X).$$

To see that this is also sufficient, set $X = \xi$. □

We define

Definition 1.2. A metric almost contact manifold admitting a characteristic connection is called a *metric almost contact manifold with torsion*.

With the above lemma we can easily prove

Theorem 1.3. *Consider a metric almost contact manifold (M, g, ψ, η). If it is of class*

1. *$\mathcal{C}_1 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7$, there exists a characteristic connection.*
2. *$\mathcal{C}_2$, $\mathcal{C}_5$, $\mathcal{C}_9$, $\mathcal{C}_{10}$, $\mathcal{C}_{11}$ or $\mathcal{C}_{12}$ there is no characteristic connection.*
3. *$\mathcal{C}_8$ there exists a characteristic connection if and only if ξ is a Killing vector field.*

Proof. We check the different cases:
In $\mathcal{C}_1$ we have $\alpha(X, X, Y) = \alpha(X, Z, \xi) = 0$ and we thus get $\alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X) = 0$.
For a structure given by α in the class $\mathcal{C}_2$ we have

$$\alpha(X, Y, Z) + \alpha(Y, Z, X) + \alpha(Z, X, Y) \;=\; \alpha(X, Y, \xi) \;=\; 0,$$

and equation (I.2) yields

$$\begin{aligned}\alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X) &= \alpha(Y, Y, \psi X) - \alpha(Y, X, \psi Y) - \alpha(X, \psi Y, Y)\\ &= \alpha(Y, Y, \psi X) + \alpha(Y, \psi Y, X) \overset{(I.1)}{=} -\alpha(Y, \psi Y, \psi^2 X) + \alpha(Y, \psi Y, X)\\ &= 2\alpha(Y, Y, \psi X).\end{aligned}$$

Thus the condition $\alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X) = 0$ implies $0 = \alpha(Y, Y, \psi^2 X) = -\alpha(Y, Y, X)$ since $\alpha(Y, Y, \xi) = 0$. Therefore α has to be also of class $\mathcal{C}_1$, which implies $\alpha = 0$.
In $\mathcal{C}_3$ we have $\alpha(X, Y, Z) = \alpha(\psi X, \psi Y, Z)$ and get

$$\begin{aligned}\alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X) &= \alpha(Y, Y, \psi X) - \alpha(\psi Y, X, Y)\\ &= \alpha(Y, Y, \psi X) - \alpha(\psi^2 Y, \psi X, Y) = \alpha(Y, Y, \psi X) + \alpha(Y, \psi X, Y) = 0\end{aligned}$$

since $\alpha(\xi, X, Y) = 0$ in $\mathcal{C}_1 \oplus ... \oplus \mathcal{C}_{10}$.
A structure is of class $\mathcal{C}_3 \oplus ... \oplus \mathcal{C}_8$ if and only if $N = 0$ thus we just have to check the condition $\alpha(Y, \xi, \psi Y) = 0$, which is satisfied in $\mathcal{C}_4$ and $\mathcal{C}_6$.
$\mathcal{C}_5$ is given by the condition $\alpha(X, Y, Z) = \frac{\delta\eta}{n-1}(F(X, Z)\eta(Y) - F(X, Y)\eta(Z))$ such that the condition $\alpha(Y, \xi, \psi Y) = 0$ implies $\delta\eta = 0$ and thus $\alpha = 0$.
For $(c, b) = (1, -1)$ in $\mathcal{C}_7$, $(c, b) = (-1, -1)$ in $\mathcal{C}_8$, $(c, b) = (1, 1)$ in $\mathcal{C}_9$ and $(c, b) = (-1, 1)$ in $\mathcal{C}_{10}$ we have

$$\alpha(X, Y, Z) \;=\; c\eta(Z)\alpha(Y, X, \xi) + b\eta(Y)\alpha(\psi X, \psi Z, \xi)$$

and get $\alpha(X, Y, \xi) = c\alpha(Y, X, \xi)$ and $\alpha(X, \psi Y, \xi) = b\alpha(X, \psi Y, \xi)$, implying $(1-cb)\alpha(Y, \psi Y, \xi) = 0$. Thus in $\mathcal{C}_7$ and $\mathcal{C}_{10}$ the vector field ξ is Killing. Since in $\mathcal{C}_7$ we have $N = 0$, we have a characteristic connection here. In $\mathcal{C}_8$ we have a characteristic connection if and only if ξ is Killing. In $\mathcal{C}_9$ and $\mathcal{C}_{10}$ we have $b = 1$ and thus

$$\begin{aligned}\alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X) &= -\eta(Y)\alpha(\psi Y, X, \xi) + c\eta(X)\alpha(Y, \psi Y, \xi) - \eta(Y)\alpha(Y, \psi X, \xi)\\ &= -2\eta(Y)\alpha(\psi Y, X, \xi) + c\eta(X)\alpha(Y, \psi Y, \xi).\end{aligned}$$

For $X = \xi$ the condition $\alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X) = 0$ implies $\alpha(Y, \psi Y, \xi) = 0$ and thus we have $0 = \alpha(\psi Y, X, \xi)$ and also $0 = \alpha(\psi^2 Y, X, \xi) = -\alpha(Y, X, \xi)$ since $\alpha(\xi, X, Y) = 0$. So we have already $\alpha = 0$.
$\mathcal{C}_{11}$ is given by the condition $\alpha(X, Y, Z) = -\eta(X)\alpha(\xi, \psi Y, \psi Z)$ and thus with $\alpha(\xi, \xi, X) = 0$ we get

$$\alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X) \;=\; \eta(Y)\alpha(\xi, \psi Y, X).$$

Because $\alpha(\xi, \psi Y, X) = 0$ already implies $\alpha(\xi, Y, X) = 0$, we obtain in this case immediately $\alpha = 0$.
In $\mathcal{C}_{12}$ we have $\alpha(X, Y, Z) = \eta(X)\eta(Y)\alpha(\xi, \xi, Z) + \eta(X)\eta(Z)\alpha(\xi, Y, \xi)$ and thus $0 = \alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X) = \eta(Y)^2\alpha(\xi, \xi, \psi X)$ gives us $\alpha = 0$. □

Remark 1.4. The conditions for a metric almost contact structure to admit a characteristic connection in Theorem 1.3 are sufficient but not necessary. In [Pu12] C. Puhle proves that in the case $n = 5$, there are structures of class $\mathcal{C}_{10} \oplus \mathcal{C}_{11}$ (in his class $\mathcal{W}_4$) carrying a characteristic connection. Thus a structure with characteristic connection is never of pure class $\mathcal{C}_{10}$ nor of class $\mathcal{C}_{11}$, but it can be of mixed class $\mathcal{C}_{10} \oplus \mathcal{C}_{11}$. But more detailed descriptions are possible in some cases. For example, if we set $Y = \xi$, the equation $0 = \alpha(Y, Y, \psi X) + \alpha(\psi Y, Y, X)$ immediately implies that a structure with characteristic connection is of class $\mathcal{C}_1 \oplus ... \oplus \mathcal{C}_{11}$.

2 Almost hermitian structures

Let (M, g) be a $2m$-dimensional Riemannian manifold equipped with a $(1, 1)$-tensor

$$J : TM \to TM \quad \text{with} \quad J^2 = -\mathrm{Id}_{TM}, \quad \text{and} \quad g(JX, JY) = g(X, Y).$$

We define a 2-form $\omega(X, Y) := g(X, JY)$. Then (M, g, J, ω) is called an almost hermitian manifold. In terms of the Levi-Civita connection ∇^g on M, the Nijenhuis tensor of M is defined to be

$$N(X, Y, Z) = g((\nabla^g_X J)(JY), Z) - g((\nabla^g_Y J)(JX), Z) + g((\nabla^g_{JX} J)(Y), Z) - g((\nabla^g_{JY} J)(X), Z).$$

Almost hermitian structures were classified by Gray and Hervella in [GH80] into four classes $\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$, which we recall in the following table.

name	class	defining relation
nearly Kähler	χ_1	$(\nabla^g_X J)X = 0$
almost Kähler	χ_2	$d\omega = 0$
balanced	χ_3	$N = 0$ and $\delta\omega = 0$
locally conformally Kähler	χ_4	$(\bar{\nabla}^g_X \omega)(Y, Z) = \frac{-1}{n-1}[g(X, Y)\delta\omega(Z) - g(X, Z)\delta\omega(Y)$ $-g(X, JY)\delta\omega(JZ) + g(X, JZ)\delta\omega(JY)]$

An almost hermitian manifold admits a characteristic connection if and only if it is of class $\chi_1 \oplus \chi_3 \oplus \chi_4$ [FI02] and it is always unique (either by explicit computation as in [FI02] or by the general Theorem 0.1). Due to the fact that in class $\chi_1 \oplus \chi_3 \oplus \chi_4$ we have $\nabla^c\omega = 0$ such manifolds are sometimes called *Kähler manifolds with torsion*, although they are evidently not Kählerian. Their characteristic torsion is given by (see for example [Ag06])

$$T \;=\; N + d\omega^J,$$

where $d\omega^J := d\omega \circ J$. For a nearly Kähler manifold (class χ_1), this connection was first introduced and investigated by A. Gray; on hermitian manifolds ($N = 0$, i. e. class $\chi_3 \oplus \chi_4$) it is sometimes called the *Bismut connection* [Bi89]. Almost hermitian manifolds of class χ_4 are locally conformally Kähler manifolds.

3 Special almost hermitian structures in dimension 6

Let (M, g) be a 6 dimensional Riemannian manifold with a 3 form ψ such that in some local basis $e_1, ..e_6$ the form ψ reads as

$$\psi = e^1 \wedge e^3 \wedge e^6 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^3 \wedge e^4 \wedge e^5.$$

Then ψ is a complex determinant corresponding to the almost hermitian structure

$$\omega = e^1 \wedge e^3 + e^1 \wedge e^4 + e^5 \wedge e^6$$

and its stabilizer is $\mathrm{SU}(3) \subset \mathrm{SO}(6)$. Such a structure can rather be defined via spinors:

The group $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ is simply connected and lifts to $\mathrm{SU}(3) \subset \mathrm{Spin}(6) \cong \mathrm{SU}(4)$. Thus an SU(3) structure can be described in terms of a stabilizing spinor. This gives a new view on SU(3) structures, which will be described in the following. This viewpoint is very useful in the hypersurface theory considered in Chapter II. To see this correspondence correctly, we need to look at the linear algebra of dimension 6.

3.1 Linear Algebra in dimension 6

We consider $\mathbb{R}^6$ and the corresponding Spin(6) representation $\Delta_6 = \Delta^+ \oplus \Delta^-$. In Δ_6 exists a real structure $\alpha : \Delta_6 \to \Delta_6$ with the following properties for all $\phi, \phi_1, \phi_2 \in \Delta_6$, $X \in \mathbb{R}^6$ and the hermitian scalar product $\langle\ ,\ \rangle$ on Δ_6 (see [Fr00], Section 1.7).

- α is real linear,
- $\alpha(i\phi) = -i\alpha(\phi)$,
- $\alpha^2 = \mathrm{Id}_\Delta$,
- α interchanges Δ^+ and Δ^-, $\alpha : \Delta^\pm \to \Delta^\mp$,
- α interchanges with the Clifford multiplication $\alpha(X\phi) = X\alpha(\phi)$ and
- $\langle \alpha(\phi_1), \phi_2 \rangle = \langle \alpha(\phi_2), \phi_1 \rangle$.

Let $\Delta := \{\phi \in \Delta_6 \mid \alpha(\phi) = \phi\}$ be the real Spin(6) representation with real scalar product $(\,.\,,\,.\,)$. Define in the Clifford algebra of $\mathbb{R}^6$ the element

$$j := e_1 \cdot \ldots \cdot e_6.$$

As an endomorphism of Δ, j satisfies $j^2 = -\mathrm{Id}$, anti commutes with the Clifford multiplication and $(j(\phi), \phi) = -(\phi, j(\phi))$. Thus j defines a Spin(6) invariant complex structure and in fact Spin(6) is isomorphic to SU(4).
Given a one dimensional subspace $V = \mathit{span}(\phi) \subset \Delta$ for any ϕ of length one we get $\dim_{\mathbb{R}}\{Y\phi \mid Y \in \mathbb{R}^6\} = 6$ since the Clifford multiplication with an element $Y \in \mathbb{R}^6$ is an isomorphism. ϕ and $j(\phi)$ are orthogonal to $X \cdot \phi$ since

$$(X \cdot \phi, j(\phi)) = -(\phi, X \cdot j(\phi)) = (\phi, j(X \cdot \phi)) = -(j(\phi), X \cdot \phi) = -(X \cdot \phi, j(\phi)).$$

This gives us the splitting

$$\Delta = \mathbb{R}\phi \oplus \mathbb{R}j(\phi) \oplus \{X \cdot \phi \mid X \in \mathbb{R}^6\}. \tag{I.4}$$

Thus we can define an orthogonal complex structure $J_\phi : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$J_\phi(X) \cdot \phi = j(X \cdot \phi).$$

We immediately get $J_\phi^2 = -\mathrm{Id}_{\mathbb{R}^6}$ and $g(JX, JY) = g(X, Y)$ for the standard metric g on $\mathbb{R}^6$. Clearly J_ϕ does not depend on the choice of $\phi \in V$.
The defining 2-form of the hermitian structure is given by

$$\omega(X, Y) := g(X, JY) = (X\phi, JY\phi) = -(X\phi, Yj(\phi)) = (\phi, XYj(\phi)) = -(j(\phi), XY\phi).$$

We can also define the 3-form ψ_ϕ via ϕ:

$$\psi_\phi(X,Y,Z) \ := \ -(X\cdot Y\cdot Z\cdot\phi\,,\,\phi)\ . \tag{I.5}$$

ψ_ϕ is invariant under J

$$\psi_\phi(JX,Y,Z)=\psi_\phi(X,JY,Z)=\psi_\phi(X,Y,JZ)$$

and thus a complex determinant. This reduces SO(6) to SU(3).

Since SU(3) is simply connected, if conversely given a reduction $\mathrm{SU}(3)\subset\mathrm{SO}(6)$ we get a lift $\mathrm{SU}(3)\subset\mathrm{Spin}(6)\cong\mathrm{SU}(4)$ fixing a complex one dimensional subspace $V\subset\Delta\cong\mathbb{C}^4$. Taking definition (I.5) together with $||\phi||=1$ as a condition, we thus get a spinor $\phi\in V$, which is unique up to ±1.

Lemma 3.1. *There is a one to one correspondence between*

- *complex structures with a complex determinant on* $\mathbb{R}^6$*,*
- *reductions of* SO(6) *to* SU(3)*,*
- *reductions of* Spin(6) *to* SU(3)*,*
- *real one dimensional subspaces of the real* Spin(6) *representation* Δ*.*

Thus the space of special hermitian structures on $\mathbb{R}^6$ *is given by*

$$\mathbb{RP}(\Delta)=\mathbb{RP}(7)=\mathrm{SO}(6)/\mathrm{SU}(3),$$

where $\mathbb{RP}(\Delta)$ *is the real projective space over the vector space* Δ*.*

We summarize some formulas expressing the action of the 2- and 3-form J_ϕ and ψ_ϕ

Lemma 3.2.

$$\psi_\phi\cdot\phi\ =\ -4\cdot\phi,\quad \psi_\phi\cdot j(\phi)\ =\ 4\cdot j(\phi),\quad \psi_\phi\cdot\phi^*\ =\ 0\quad \textit{if}\ \ \phi^*\perp\phi,j(\phi),$$
$$(X\lrcorner\,\psi_\phi)\cdot\phi\ =\ 2\,X\cdot\phi\quad X\in\mathbb{R}^6,\quad J_\phi\cdot\phi\ =\ 3\,j(\phi),\quad J_\phi\cdot j(\phi)\ =\ -3\,\phi.$$

3.2 SU(3) manifolds

An SU(3) manifold (M^6,g,ϕ) is a Riemannian spin manifold equipped with a global spinor ϕ of length one. We always denote its spinor bundle by Σ and its Levi-Civita connection by ∇. The induced SU(3)-structure is given by the 3-form ψ_ϕ. Let $\omega=g(.,J.)$ be the hermitian 2-form defining the corresponding U(3)-structure. We define a second 3-form by

$$\psi^J_\phi(X,Y,Z):=\psi_\phi(JX,JY,JZ)=-\psi_\phi(JX,Y,Z)=-(XYZ\phi,j(\phi)).$$

We shall recover the various SU(3)-types essentially by reinterpreting the intrinsic torsion. With the splitting (I.4) we have

$$\nabla\phi=\eta\otimes j(\phi)+S\otimes\phi,$$

for some 1-form η and a linear map $S\in\mathrm{End}(TM^6)$. Moreover we have

Lemma 3.3. *S and η are given by*

$$(\nabla_X\omega)(Y,Z)=2\psi^J_\phi(S(X),Y,Z)\qquad \textit{and}\qquad 8\eta(X)=-(\nabla_X\psi^J_\phi)(\psi_\phi)$$

for any X,Y,Z.

Proof. We immediately find $\eta = (\nabla\phi, j(\phi))$. Since j is the volume form, it is parallel under ∇ and we conclude

$$\nabla_X(j(\phi)) = j\nabla_X\phi = jS(X)\phi + j\eta(X)j(\phi) = -S(X)j(\phi) - \eta(X)\phi.$$

With $\omega(X,Y) = -(X\phi, Yj(\phi))$ we get

$$\begin{aligned}
-(\nabla_X\omega)(Y,Z) &= X(Y\phi, Zj(\phi)) - (\nabla_X Y\phi, Zj(\phi)) - (Y\phi, \nabla_X Zj(\phi)) \\
&= (\nabla_X(Y\phi), Zj(\phi)) + (Y\phi, \nabla_X(Zj(\phi))) - (\nabla_X Y\phi, Zj(\phi)) - (Y\phi, \nabla_X Zj(\phi)) \\
&= (Y\nabla_X\phi, Zj(\phi)) + (Y\phi, Z\nabla_X j(\phi)) \\
&= (YS(X)\phi, Zj(\phi)) - (Y\phi, ZS(X)j(\phi)) - \eta(X)(Y\phi, Z\phi) + \eta(X)(Yj(\phi), Zj(\phi)) \\
&= -(ZYS(X)\phi, j(\phi)) - (S(X)ZY\phi, j(\phi)) \\
&= \psi_\phi^J(Z,Y,S(X)) + \psi_\phi^J(S(X),Z,Y) \\
&= \psi_\phi^J(Y,S(X),Z) - \psi_\phi^J(S(X),Y,Z) \\
&= -2\psi_\phi^J(S(X),Y,Z)
\end{aligned}$$

Furthermore, the computation

$$\begin{aligned}
\nabla_X(\psi_\phi^J)(\psi_\phi) &= -X(\psi_\phi\phi, j(\phi)) + (\nabla_X\psi_\phi\phi, j(\phi)) \\
&= -(\psi_\phi\nabla_X\phi, j(\phi)) - (\psi_\phi\phi, \nabla_X j(\phi)) \\
&= -(\psi_\phi S(X)\phi, j(\phi)) + (\psi_\phi\phi, S(X)j(\phi)) \\
&\quad -\eta(X)(\psi_\phi j(\phi), j(\phi)) + \eta(X)(\psi_\phi\phi, \phi) \\
&= 2\eta(X)(\psi_\phi\phi, \phi) = -8\eta(X)
\end{aligned}$$

finishes the proof. □

To better understand the role of the pair (S,η) we will work with the SU(3)-connection

$$\nabla^n_X Y = \nabla_X Y - \Gamma(X)(Y),$$

given by the Levi-Civita connection ∇ minus the intrinsic torsion Γ, see [Sa89]. Decompose $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{m}$, then Γ is a one form with values in $\mathfrak{m}$. We shall repeatedly use one symbol for covariant derivatives on the tangent bundle and lifted covariant derivatives on the spin bundle, hence

$$\nabla^n_X\phi^* = \nabla_X\phi^* - \tfrac{1}{2}\Gamma(X)\phi^*$$

for any spinor ϕ^*.

Proposition 3.4. *The intrinsic torsion of the* SU(3)*-structure* (M^6, g, ϕ) *is given by*

$$\Gamma = S\lrcorner\psi_\phi - \frac{2}{3}\eta\otimes\omega$$

where $S\lrcorner\psi_\phi(X,Y,Z) := \psi_\phi(S(X),Y,Z)$.

Proof. The spinor ϕ is parallel for ∇^n, as $\mathrm{Stab}(\phi) = \mathrm{SU}(3)$, so $\nabla_X\phi = \frac{1}{2}\Gamma(X)\phi$. By Lemma 3.2 we have $\omega\phi = -3j(\phi)$, so

$$\nabla_X\phi = S(X)\phi + \eta(X)j(\phi) = \frac{1}{2}(S(X)\lrcorner\psi_\phi)\phi - \frac{1}{3}\eta(X)\omega\phi.$$

Since $(X\lrcorner\psi_\phi)(Y, J_\phi Z) = (X\lrcorner\psi_\phi)(J_\phi Y, Z)$ we see that $X\lrcorner\psi_\phi \in \mathfrak{su}(3)^\perp$ and since $\omega \in \mathfrak{su}(3)^\perp$, the 1-form $S\lrcorner\psi_\phi - \frac{2}{3}\eta\otimes\omega$ is the intrinsic torsion of the spinorial connection: Suppose

$$\Gamma = \tilde{\Gamma} + r$$

for some r with values in $\mathfrak{su}(3)^\perp$. Then we have

$$\tilde{\Gamma}(X)\phi = 2\nabla_X\phi = \Gamma(X)\phi = \tilde{\Gamma}(X)\phi + r(X)\phi$$

and thus $r(X)\phi = 0$. Since the stabilizer of ϕ is SU(3) this implies $r(X) \in \mathfrak{su}(3)$ and with the assumption $r(X) \in \mathfrak{su}(3)^\perp$ we get $\Gamma = \tilde{\Gamma}$. ☐

In the light of this fact we may call S the *intrinsic endomorphism* and η the *intrinsic* 1-*form*, for reference.
The classification of the SU(3) structure is given by η and S and we consider the space $TM \oplus End(TM)$ of all such forms. Under SU(3) we have the decomposition

$$\begin{array}{rcl} \mathrm{End}(\mathbb{R}^6) & = & \mathbb{R}\cdot J \ \oplus\ \mathbb{R}\cdot \mathrm{Id} \ \oplus\ \mathfrak{su}(3) \\ & \oplus & \{A \in S^2_0(\mathbb{R}^6) | AJ = JA\} \\ & \oplus & \{A \in S^2_0(\mathbb{R}^6) | AJ = -JA\} \\ & \oplus & \{A \in \Lambda^2(\mathbb{R}^6) | AJ = -JA\} \end{array}$$

with dimensions $36 = 1 + 1 + 8 + 8 + 12 + 6$. We compare those to the classes of special almost hermitian structures $\chi_1^- \oplus \chi_1^+ \oplus \chi_2^- \oplus \chi_2^+ \oplus \chi_3 \oplus \chi_4 \oplus \chi_5$ given in [CS02]. An SU(3) structure induces an U(3) structure. This U(3) structure is of type $\chi_{i_1} \oplus .. \oplus \chi_{i_k}$ for $1 \leqq i_1 < .. < i_k \leqq 4$ in the Gray-Hervella classification [GH80] if and only if the SU(3) structure is of type $\chi_{i_1} \oplus .. \oplus \chi_{i_k} \oplus \chi_5$. Thus an SU(3) structure is of type χ_5 if the U(3) structure is Kähler and 1-form η determines the class χ_5. Comparing the dimensions of the other SU(3) modules one can identify the classes. We only need a closer look at χ_i^+ and the χ_i^- parts for $i = 1, 2$. It suffices to look at Example 3.12 or at Remark 3.7. In the following we use

Notation 3.5. From now on we will write $\chi_{1\bar{1}2\bar{2}345}$ for $\chi_1^+ \oplus \chi_1^- \oplus \chi_2^+ \oplus \chi_2^- \oplus \chi_3 \oplus \chi_4 \oplus \chi_5$ and denote subspaces in the obvious way, for example $\chi_1^+ \oplus \chi_2^- \oplus \chi_4$ will be written as $\chi_{1\bar{2}4}$. Thus χ_1^+ and χ_1^- will be denoted by χ_1 and $\chi_{\bar{1}}$.

Lemma 3.6. *The classes of an* SU(3) *structure* (M^6, g, ϕ) *are determined as follows.*

class	*description*	*dimension*
χ_1	$S = \lambda \cdot J_\phi$, $\eta = 0$	1
$\chi_{\bar{1}}$	$S = \mu \cdot \mathrm{Id}$, $\eta = 0$	1
χ_2	$S \in \mathfrak{su}(3)$, $\eta = 0$	8
$\chi_{\bar{2}}$	$S \in \{A \in S^2_0(\mathbb{R}^6) \| AJ_\phi = J_\phi A\}$, $\eta = 0$	8
χ_3	$S \in \{A \in S^2_0(\mathbb{R}^6) \| AJ_\phi = -J_\phi A\}$, $\eta = 0$	12
χ_4	$S \in \{A \in \Lambda^2(\mathbb{R}^6) \| AJ_\phi = -J_\phi A\}$, $\eta = 0$	6
χ_5	$S = 0$, $\eta \neq 0$	6

where $\lambda, \mu \in \mathbb{R}$. *In particular* S *is symmetric and* $\eta = 0$ *if and only if the type is* $\chi_{\bar{1}\bar{2}3}$.

Remark 3.7. An SU(3) structure is half flat (of type $\chi_1^- \oplus \chi_2^- \oplus \chi_3$), if the manifold M is possibly a submanifold of a manifold $\bar{M}$ with holonomy contained in G_2 (see [CS02]). The G_2 structure on $\bar{M}$ thus is given by a parallel spinor.
On the other hand we showed that being half flat means $\nabla_X\phi = S(X)\phi$ with some symmetric S. Thus ϕ can possibly be lifted to a flat spinor on a manifold $\bar{M}$ with Weingarten map S (see [BGM05]), defining a flat G_2 structure and the two statements are the same.

3.3 Spinorial characterisation

The description of SU(3)-structures in terms of ϕ is the main result of this section. To start with we will describe some geometric quantities of the SU(3) structure and their correspondence to ϕ. Denote by D the Riemannian Dirac operator.

Lemma 3.8. *On an* SU(3) *manifold the* χ_4*-component of the intrinsic torsion is given by*

$$\delta\omega(X) = 2[(D\phi, Xj(\phi)) - \eta(X)].$$

In particular $\delta\omega = 0$ *is equivalent to* $(D\phi, Xj(\phi)) = \eta(X)$*. The Lee form is given by*

$$\theta(X) = \delta\omega \circ J(X) = 2(D\phi, X\phi) - 2\eta \circ J(X).$$

Proof. We have

$$(\nabla_X\omega)(Y,Z) = (ZY\nabla_X\phi, j(\phi)) + (ZY\phi, \nabla_X j(\phi)) = -2(YZ\nabla_X\phi, j(\phi)) - 2g(Y,Z)\eta(X),$$

leading to

$$\begin{aligned}\delta\omega(X) &= -\sum_i(\nabla_{e_i}\omega)(e_i,X) = \sum_i(\nabla_{e_i}\omega)(X,e_i)\\ &= -2\sum_i((Xe_i\nabla_{e_i}\phi, j(\phi)) - g(X,e_i)\eta(e_i))\\ &= -2(XD\phi, j(\phi)) - 2\eta(X) = 2(D\phi, Xj(\phi)) - 2\eta(X).\end{aligned}$$ □

We consider the space of all possible types of structures $T^*M^6 \otimes \phi^\perp \ni \nabla\phi$, where $\phi^\perp = \mathbb{R}j(\phi) \oplus \{X\cdot\phi \mid X \in TM^6\}$ is the orthogonal complement of ϕ in Σ. The restricted Clifford multiplication m is defined by

$$m: \ T^*M^6 \otimes \phi^\perp \to \Sigma.$$

Let $\pi : \mathrm{Spin}(6) \to \mathrm{SO}(6)$ be the usual projection. For $h \in \mathrm{Spin}(6)$ we have

$$m(\pi(h)\eta \otimes h\phi^*) = h\eta h^{-1}h\phi^* = hm(\eta\otimes\phi^*)$$

and m is Spin(6) equivariant and thus SU(3) equivariant. Comparing the dimensions of the modules given in (I.4) and the ones of Lemma 3.6 we see that $\chi_{2\bar{2}3} \subset Ker(m)$ and with

$$D\phi = 6\lambda j(\phi) \text{ for } S = \lambda J_\phi \text{ and } D\phi = -6\mu\phi \text{ for } S = \mu\mathrm{Id}$$

we have the correspondences

$$\chi_1 \to \mathbb{R}j(\phi) \text{ and } \chi_{\bar{1}} \to \mathbb{R}\phi$$

and get $(D\phi, j(\phi)) = 6\lambda$ and $(D\phi,\phi) = -6\mu$. For a closer look at χ_{45} we recall that $\{J_\phi e_i\phi, \phi, j(\phi)\}$ is a basis of Σ for some local orthonormal frame e_i, hence

$$D\phi = \sum_{i=1}^{6}(D\phi, J_\phi e_i\phi)J_\phi e_i\phi + (D\phi,\phi)\phi + (D\phi, j(\phi))j(\phi).$$

With Lemma 3.8 we conclude

$$D\phi = \sum_{i=1}^{6}[\frac{1}{2}\delta\omega(e_i) + \eta(e_i)]e_ij(\phi) + 6\lambda j(\phi) - 6\mu\phi = (\frac{1}{2}\delta\omega + \eta)j(\phi) + 6\lambda j(\phi) - 6\mu\phi.$$

Thus as the image of the Clifford multiplication, the $\mathbb{R}^6$ component of Σ^- is determined by $\delta\omega + 2\eta$.

Theorem 3.9. *On a 6-dimensional Riemannian spin manifold there exists a spinor $\phi \in \Sigma$ of length one in the kernel of the Dirac operator*

$$D\phi = 0$$

if and only if it admits an SU(3) *structure of type $\chi_{2\bar{2}345}$ with the restriction $\delta\omega = -2\eta$ on the χ_4 and the χ_5 part of the intrinsic torsion.*
The χ_1- and the $\chi_{\bar{1}}$-component of the intrinsic torsion are given by

$$(D\phi, j(\phi)) = 6\lambda \text{ and } (D\phi, \phi) = -6\mu.$$

Remark 3.10. One could also look at the Spin(6) invariant Twistor operator P restricted to $\Sigma^+ \supset i\mathbb{R}\phi \oplus \phi^\perp$, the projection on the kernel of the nonrestricted Dirac operator. But since the (nonrestricted) Dirac operator mixes the χ_4 part with the $TM \otimes \phi$ part, this projection is not useful to us.

As one can see in Lemma 3.6, the $\chi_{1\bar{1}}$ part are also determined by the trace of $J_\phi S$ and S. Indeed we have

$$\mathrm{tr}(S) = -(D\phi, \phi) = 6\mu \text{ and } \mathrm{tr}(J_\phi S) = -(D\phi, j(\phi)) = -6\lambda.$$

The linear combination $\chi_4 + 2\chi_5$ vanishing in the theorem also shows up (up to volume choice) in work of Cardoso *et al.* [CCD03] and plays a role in supersymmetric compactifications of heterotic string theory.

Example 3.11. Consider the Lie algebra $\mathfrak{g} = \mathrm{span}\{e_1, \ldots, e_6\}$ with structure equations

$$de^i = 0 \text{ if } i = 3,4,5 \text{ and } de^1 = e^3 \wedge e^4 + 2e^3 \wedge e^5,\ \ de^2 = e^4 \wedge e^5,\ \ de^6 = e^5 \wedge e^1 + e^2 \wedge e^3.$$

Since the structure constants are rational the corresponding 1-connected Lie group G has a co-compact lattice Γ. We consider the spin structure on $M^6 = G/\Gamma$ given by the pointwise construction of the Clifford algebra of the global vector fields e_i given above and the SU(3) structure determined by choosing $\phi = (1,0,0,0,0,0,0,1)^t$. With $g(\nabla_{e_i} e_j, e_k) = -de_k(e_i, e_j) - de^j(e_k, e_i) + de^i(e_j, e_k)$ this gives us

$$S = \frac{1}{2}\begin{pmatrix} 0 & 0 & & & & \\ -2 & 0 & & & & \\ & & 0 & 1 & -1 & 0 \\ & & 1 & 0 & 0 & -1 \\ & & -1 & 0 & 0 & -1 \\ & & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \eta = e_1$$

and it is not hard to see that $D\phi = 0$. The structure has type $\chi_{2\bar{2}345}$, and the presence of the components 4 and 5 is reflected in the non-vanishing η.

Example 3.12. We look at *further example 3* of section 4 of [CS02]: The nilpotent 3-step Lie algebra given by

$$de^i = 0, \text{ if } i = 1,2,4,5; \quad de^3 = e^2 \wedge e^5 \text{ and } de^6 = e^1 \wedge e^4 - e^2 \wedge e^3.$$

Again we calculate ∇ and get

$$\nabla_{e_1} = -E_{46}, \nabla_{e_2} = E_{35} + E_{36}, \nabla_{e_3} = E_{25} - E_{26}, \nabla_{e_4} = E_{16}, \nabla_{e_5} = E_{23}, \nabla_{e_6} = E_{14} - E_{23}$$

where $E_{ij}e_i = e_j$ and $E_{ij}e_j = -e_i$. With the lift $\frac{1}{2}e_ie_j$ of E_{ij} one gets

$$S = \frac{1}{2}\begin{pmatrix} 0 & -1 & & & & \\ -1 & -1 & & & & \\ & & 1 & 1 & & \\ & & 1 & 0 & & \\ & & & & -1 & \\ & & & & & 0 \end{pmatrix}$$

and $\eta = 0$. This is a typical example of a half flat SU(3) structure.

Notation 3.13. We decompose the intrinsic endomorphism into

$$S = \lambda J_\phi + \mu \mathrm{Id} + S_2 + S_{34}$$

where $J_\phi S_2 = S_2 J_\phi$, $J_\phi S_{34} = -S_{34} J_\phi$ and the trace of S_2 and $J_\phi S_2$ vanishes.

We shall now prove that the Niejenhuis tensor determines the $\chi_{1\bar{1}2\bar{2}}$-component in the following way.

Lemma 3.14.

$$N(X,Y,Z) = -2[\psi_\phi^J((J_\phi \circ S + S \circ J_\phi)X, Y, Z) - \psi_\phi^J((J_\phi \circ S + S \circ J_\phi)Y, X, Z)].$$

Thus if the structure is of type $\chi_{1\bar{1}345}$, the Niejenhuis tensor is given by

$$N(X,Y,Z) = 8[\lambda \psi_\phi^J(X,Y,Z) - \mu \psi_\phi(X,Y,Z)].$$

Proof. We have $g((\nabla_X J_\phi)Y, Z) = -(\nabla_X \omega)(Y,Z)$ and get with Lemma 3.3

$$\begin{aligned} N(X,Y,Z) &= -(\nabla_X\omega)(J_\phi Y, Z) + (\nabla_Y \omega)(J_\phi X, Z) - (\nabla_{J_\phi X}\omega)(Y,Z) + (\nabla_{J_\phi Y}\omega)(X,Z) \\ &= 2[-\psi_\phi^J(SX, J_\phi Y, Z) + \psi_\phi^J(SY, J_\phi X, Z) - \psi_\phi^J(SJ_\phi X, Y, Z) + \psi_\phi^J(SJ_\phi Y, X, Z)] \\ &= 2[-\psi_\phi^J(J_\phi SX, Y, Z) + \psi_\phi^J(J_\phi SY, X, Z) - \psi_\phi^J(SJ_\phi X, Y, Z) + \psi_\phi^J(SJ_\phi Y, X, Z)] \\ &= 2[-\psi_\phi^J((J_\phi S + SJ_\phi)X, Y, Z) + \psi_\phi^J((J_\phi S + SJ_\phi)Y, X, Z)]. \end{aligned}$$

Furthermore for $S = \lambda J_\phi + \mu\mathrm{Id} + S_{34}$ we have

$$J_\phi S + SJ_\phi = J_\phi(S_{34} + \lambda J_\phi + \mu \mathrm{Id}) + (S_{34} + \lambda J_\phi + \mu\mathrm{Id})J_\phi = -2\lambda \mathrm{Id} + 2\mu J_\phi.$$

and get

$$\begin{aligned} N(X,Y,Z) &= 2[-\psi_\phi^J((-2\lambda\mathrm{Id} + 2\mu J_\phi)X, Y, Z) + \psi_\phi^J((-2\lambda \mathrm{Id} + 2\mu J_\phi)Y, X, Z)] \\ &= 4[\psi_\phi^J((\lambda\mathrm{Id} - \mu J_\phi)X, Y, Z) + \psi_\phi^J((-\lambda\mathrm{Id} + \mu J_\phi)Y, X, Z)] \\ &= 4[\lambda\psi_\phi^J(X,Y,Z) - \mu\psi_\phi^J(J_\phi X, Y, Z) - \lambda\psi_\phi^J(Y,X,Z) + \mu\psi_\phi^J(J_\phi Y, X, Z)] \\ &= 8[\lambda\psi_\phi^J(X,Y,Z) - \mu\psi_\phi(X,Y,Z)]. \end{aligned}$$

This proves the claim. □

The $\chi_{1\bar{1}34}$ part of the intrinsic torsion is given by $d\omega$ in the following way:

Lemma 3.15. *With Notation 3.13 we have*

$$d\omega(X,Y,Z) = 6\lambda\psi_\phi(X,Y,Z) + 6\mu\psi_\phi^J(X,Y,Z) + 2 \overset{XYZ}{\mathfrak{S}} \psi_\phi^J(S_{34}(X), Y, Z).$$

Proof. We have $d\omega(X,Y,Z) = \overset{XYZ}{\mathfrak{S}} (\nabla_X\omega)(Y,Z)$, The fact that $\overset{XYZ}{\mathfrak{S}} \psi_\phi^J(S_2(X), Y, Z)$ vanishes corresponds to $d\omega = 0$ in the class $\chi_{2\bar{2}5}$. □

To get additional equations in terms of the corresponding spinor ϕ, we need the following lemma.

Lemma 3.16. *The intrinsic torsion (S, η) of a Riemannian spin manifold (M^6, g, ϕ) satisfies the following properties:*

$$\begin{aligned} SJ_\phi = J_\phi S \quad &\Longleftrightarrow \quad (J_\phi Y \nabla_X \phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi), \\ SJ_\phi = -J_\phi S \quad &\Longleftrightarrow \quad (J_\phi Y\nabla_X\phi, \phi) = (Y\nabla_{J_\phi X}\phi, \phi), \\ S \text{ is symmetric} \quad &\Longleftrightarrow \quad (X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi), \\ S \text{ is skew-symmetric} \quad &\Longleftrightarrow \quad (X\nabla_Y\phi,\phi) = -(Y\nabla_X\phi, \phi). \end{aligned}$$

Proof. With $(J_\phi S(X)\phi, Y\phi) = (SJ_\phi(X)\phi, Y\phi)$ if and only if

$$(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi(X)}\phi, \phi),$$

we prove the first two equivalences. Since $\phi, j(\phi) \perp Y\phi$ for any $Y \in TM^6$, we obtain

$$g(S(X), Y) = (\nabla_X\phi, Y\phi) \quad \text{and} \quad g(X, S(Y)) = (\nabla_Y\phi, X\phi).$$

Thus we conclude the formulas for symmetric and skew symmetric S. □

Theorem 3.17. *The classification of* SU(3) *structures in terms of the defining spinor ϕ is given in the following table, where η is the one-form given by $\eta(X) := (\nabla_X\phi, j(\phi))$ and the functions λ and μ are given by $\frac{1}{6}(D\phi, j(\phi))$ and $-\frac{1}{6}(D\phi, \phi)$.*

class	*spinorial equation*
χ_1	$\nabla_X\phi = \lambda Xj(\phi)$ *for* $\lambda \in \mathbb{R}$
$\chi_{\bar{1}}$	$\nabla_X\phi = \mu X\phi$ *for* $\mu \in \mathbb{R}$
χ_2	$(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$, $(Y\nabla_X\phi, j(\phi)) = (X\nabla_Y\phi, j(\phi))$, $\lambda = \eta = 0$
$\chi_{\bar{2}}$	$(J_\phi Y\nabla_X\phi, \phi) = (Y\nabla_{J_\phi X}\phi, \phi)$, $(Y\nabla_X\phi, j(\phi)) = -(X\nabla_Y\phi, j(\phi))$, $\mu = \eta = 0$
χ_3	$(J_\phi Y\nabla_X\phi, \phi) = (Y\nabla_{J_\phi X}\phi, \phi)$, $(Y\nabla_X\phi, j(\phi)) = (X\nabla_Y\phi, j(\phi))$, *and* $\eta = 0$
χ_4	$(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$, $(Y\nabla_X\phi, j(\phi)) = -(X\nabla_Y\phi, j(\phi))$ *and* $\eta = 0$
χ_5	$\nabla_X\phi = (\nabla_X\phi, j(\phi))j(\phi)$
$\chi_{1\bar{1}}$	$\nabla_X\phi = \lambda Xj(\phi) + \mu X\phi$
$\chi_{2\bar{2}}$	$(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$, $\lambda = \mu = 0$ *and* $\eta = 0$
$\chi_{2\bar{2}5}$	$(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$ *and* $\lambda = \mu = 0$
$\chi_{1\bar{1}2\bar{2}}$	$(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$ *and* $\eta = 0$
$\chi_{1\bar{1}2\bar{2}5}$	$(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$
$\chi_{2\bar{2}3}$	$D\phi = 0$ *and* $\eta = 0$
$\chi_{1\bar{1}2\bar{2}3}$	$(D\phi, X\phi) = 0$ *and* $\eta = 0$
$\chi_{1\bar{1}2\bar{2}34}$	$(\nabla_X\phi, j(\phi)) = 0$
$\chi_{2\bar{2}35}$	$(D\phi, Xj(\phi)) = \eta(X)$ *and* $\lambda = \mu = 0$
$\chi_{1\bar{1}2\bar{2}35}$	$(D\phi, Xj(\phi)) = \eta(X)$
χ_{34}	$(J_\phi Y\nabla_X\phi, \phi) = (Y\nabla_{J_\phi X}\phi, \phi)$ *and* $\eta = 0$
χ_{345}	$(J_\phi Y\nabla_X\phi, \phi) = (Y\nabla_{J_\phi X}\phi, \phi)$
$\chi_{2\bar{2}345}$	$\lambda = \mu = 0$
$\chi_{\bar{1}\bar{2}3}$	$(X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi)$ *and* $\eta = 0$

Proof. We first prove that λ and μ in χ_1 and $\chi_{\bar{1}}$ are constant. In χ_1 we have $S = \lambda J_\phi$ and thus $\nabla_X(\phi + j(\phi)) = -\lambda X(\phi + j(\phi))$. Since a nearly Kähler structure (type $\chi_{1\bar{1}5}$) is given by a Killing spinor [Gr90], this λ must be constant. In the case $\chi_{\bar{1}}$ the spinors ϕ and $j(\phi)$ themselves are

Killing spinors with Killing constants μ and $-\mu$.

We combine the results of Lemma 3.16 as follows. By Lemma 3.6, a structure is of type χ_2 if S is skew symmetric, commutes with J_ϕ, $J_\phi S = SJ_\phi$ and the trace of $J_\phi S$ and η vanish. With Lemma 3.16 we get the condition $(X\nabla_Y\phi,\phi) = -(Y\nabla_X\phi,\phi)$ under which the equation $(J_\phi Y\nabla_X\phi,\phi) = -(Y\nabla_{J_\phi X}\phi,\phi)$ is equivalent to

$$(Y\nabla_X\phi, j(\phi)) = -(J_\phi Y\nabla_X\phi,\phi) = (X\nabla_{J_\phi Y}\phi,\phi) = -(J_\phi X\nabla_Y\phi,\phi) = (X\nabla_Y\phi, j(\phi)).$$

The other classes are to be calculated similarly, making extensive use of Lemmas 3.8, 3.16. □

It makes little sense to compute all possible combinations (in principle, 2^7), so we have listed here only some that raise interest. Others can be inferred by arguments of the following sort: suppose we want to show that class χ_{124} has $(X\nabla_Y\phi,\phi) = -(Y\nabla_X\phi,\phi)$ and $\eta = 0$ as defining equations. From Lemma 3.6 we know χ_{124} is governed by the fact that S is skew symmetric, and at the same time η controls χ_5, whence the claim.
Other identities are given by the following kind of argument. Assume we want to show that the equation

$$\overset{XYZ}{\mathfrak{S}}(YZ\nabla_X\phi, j(\phi)) + \overset{XYZ}{\mathfrak{S}}\eta(X)g(Y,Z) = 3\lambda\psi_\phi(X,Y,Z) + 3\mu\psi_\phi^J(X,Y,Z)$$

defines the class $\chi_{1\bar{1}2\bar{2}5}$. From Lemma 3.15 we know that $d\omega = 6\lambda\psi_\phi(X,Y,Z) + 6\mu\psi_\phi^J(X,Y,Z)$ defines this class, so we finish with $d\omega(X,Y,Z) = \overset{XYZ}{\mathfrak{S}}(\nabla_X\omega)(Y,Z)$ and the first equality in the proof of Lemma 3.8.
Additionally to $d\omega$ one can compute $d\psi_\phi$ and $d\psi_\phi^J$ in terms of ϕ and then use the correspondences in [CS02] to get more equations (e.g. a structure is half flat, of type $\chi_{\bar{1}\bar{2}3}$, if $d\omega\wedge\omega = 0$ and $d\psi_\phi^J = 0$).

Remark 3.18. (i) The proof shows that the real Killing spinors of an SU(3)-structure of type $\chi_{1\bar{1}}$ (with Killing constants $\pm|\lambda|$) have the form

$$\phi \pm j(\phi) \text{ in the case } \chi_1 \text{ and } \phi \text{ and } j(\phi) \text{ in the case } \chi_{\bar{1}}$$

Now, in class $\chi_{\bar{1}23}$ we have the constraint $D\phi = f\phi$, so ϕ is an eigenspinor with eigenfunction f. (One can change ϕ such that one achieves an eigenspinor even in $\chi_{1\bar{1}2\bar{2}3}$.) But we are not aware of a nice argument, showing that f is constant as in the nearly Kähler case.
(ii) Rescaling ϕ to $f_1\phi + f_2 j(\phi)$ by functions f_1, f_2 with $f_1^2 + f_2^2 = 1$ affects the structure as follows: the intrinsic tensors transform as $S \rightsquigarrow (f_1^2 - f_2^2)S + 2f_1f_2J_\phi\circ S$ and $\eta \rightsquigarrow \eta + \frac{df_2}{f_1}$. (cf. Section 1.1 for the case where $f = h$ is constant on M^6). The χ_5-component varies, and $\chi_i^\pm, i = 1, 2$ change, too, cf. [CS02]. Therefore, if we are looking at a Killing spinor (corresponding to SU(3)-type $\chi_{1\bar{1}5}$), then f necessarily determines the fifth component $\eta = -\frac{df_2}{f_1}$.
(iii) It is fairly evident (cf. [CS02]) that the effect of a rotation $S \mapsto JS$ is the exchange $\chi_j^+ \longleftrightarrow \chi_j^-$, $j = 1, 2$, while the other ones are untouched.

Example 3.19. Schoemann describes in [Sc06] the almost complex structures on the twistor space $\mathbb{CP}^3$ over the manifold S^4. He uses the construction of [BFGK91, Section 3.3] to consider a family of metrics g_t on $\mathbb{CP}^3$ given by

$$g_t := \pi^*\hat{g} + t\tilde{g},$$

where $\pi^*\hat{g}$ is the metric on S^4 beeing pulled back by $\pi : \mathbb{CP}^3 \to \mathcal{S}^4$ and $(\tilde{g}, \tilde{J})$ is the standard Kähler structure on the fibre S^2 over a point in S^4. One defines an almost complex structures J over a point J_x in $\mathbb{CP}^3$ by

$$J = \pi^{-1}\circ J_x\circ\pi - \tilde{J}.$$

Schoemann describes the types of the structure in dependence of t. The type of the corresponding SU(3) structure is $\chi_{1\bar{1}2\bar{2}5}$ for general t and $\chi_{1\bar{1}}$ for $t = \frac{1}{2}$. As the homogenous space $SO(5)/U(2)$, this structure is given by a invariant spinor ϕ. In particular the general equation for such a structure is

$$(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi)$$

for general t and the Killing equation for $t = \frac{1}{2}$. We continue the calculation of [BFGK91] and get

$$\nabla_X \phi = S(X)\phi \text{ where } S = diag(\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{1-t}{2\sqrt{t}}, \frac{1-t}{2\sqrt{t}})$$

in particular $\eta = 0$. There is another structure J^+ on $\mathbb{CP}^3$, which is Kähler at the value $t = 1$. Unfortunately there is no invariant spinor in our description of $\mathbb{CP}^3$ as $SO(5)/U(2)$ describing J^+.

In the same way, one could look at the Flag manifold $F_{1,2} = \mathrm{U}(3)/\mathrm{U}(1)^3$ as the twistor space of $\mathbb{CP}^2$. In this case we know from [AGI98] that all the complex structure are invariant.

3.4 Adapted connections

Let (M^6, g, ϕ) be a 6 dimensional SU(3) manifold with Levi-Civita connection ∇.

As we are interested in non integrable structures, $\nabla\phi \neq 0$, we are looking for a metric connection that preserves the SU(3) structure.

In Section 3.2 we introduced the Levi-Civita connection $Z = \tilde{Z} + \Gamma$, where Γ is the intrinsic torsion. We will consider the SU(3)-connection $\tilde{Z}$. This connection is often called *canonical connection*.

The space of all metric connections $\bar{\nabla}$ is isomorphic to the space of all $(2,1)$ tensors $\mathcal{A}^g := A \in TM^6 \otimes \Lambda^2(TM^6)$ by $\tilde{\nabla}_X Y = \nabla_X Y + A(X,Y)$. Let S be the intrinsic endomorphism and η the intrinsic 1-form of the SU(3) structure on M^6. We define the map

$$\Xi : TM^6 \oplus \mathrm{End}(TM^6) \to \mathcal{A}^g,\ (\eta, S) \mapsto -S \lrcorner \psi_\phi + \frac{2}{3}\eta \otimes \omega.$$

Then $\nabla^n_X Y := \nabla_X Y + \Xi(\eta, S)$ defines a metric connection on M^6 and with Lemma 3.2 we get

$$\begin{aligned} \nabla^n_X \phi &= \nabla_X \phi - \psi_\phi \frac{1}{2}(S(X), ., .) \cdot \phi + \frac{1}{3}\eta(X)\omega \cdot \phi \\ &= S(X) \cdot \phi + \eta(X) j(\phi) - S(X) \cdot \phi - \eta(X) j(\phi) = 0. \end{aligned}$$

Thus ∇^n is an SU(3) connection. The space $\mathcal{A}^g$ splits under the representation of SO(6) in (see page 51 of [Ca25] and [AF04])

$$\mathcal{A}^g = TM^6 \oplus \Lambda^3(TM^6) \oplus \mathcal{T}.$$

The three classes are given in the following table

class	name	relation
TM	vectorial torsion	$\exists V \in TM$ s.t. $A(X,Y,Z) =$ $g(X,Y)g(Z,V) - g(X,Z)g(Y,V)$
$\Lambda^3(TM)$	(totally) skew symmetric torsion	$A(X,Y,Z) = -A(Y,X,Z)$
$\mathcal{T}$	cyclic traceless torsion	$\overset{XYZ}{\mathfrak{S}}\ A(X,Y,Z) = 0$ and $\sum_i A(e_i, e_i, X) = 0$

for some local basis e_1 with $i = 1..7$ and $X, Y, Z \in TM$. A metric connection is said to be of a certain type, if its torsion is of this type.
With an appropriate computer algebra system one calculates the map Ξ at one point and gets

Lemma 3.20. *On a* 6*-dimensional* SU(3) *manifold the type of the connection* ∇^n *is given in the following table.*

type of M^6	$\chi_{1\bar{1}}$	$\chi_{2\bar{2}}$	χ_3	χ_4	χ_5
type of ∇^n	$\Lambda^3(\mathbb{R}^6)$	$\mathcal{T}$	$\Lambda^3(\mathbb{R}^6) \oplus \mathcal{T}$	$\mathbb{R}^6 \oplus \Lambda^3(\mathbb{R}^6) \oplus \mathcal{T}$	$\mathbb{R}^6 \oplus \Lambda^3(\mathbb{R}^6) \oplus \mathcal{T}$

The projection to the skew symmetric part of the torsion given in the previous lemma gives the characteristic connection of the SU(3) structure.
We are interested in finding out whether and when an SU(3) structure (M^6, g, ϕ) admits a characteristic connection. Any such must be the (unique) characteristic connection of the underlying U(3)-structure with the additional condition $\nabla^c \psi_\phi = 0$ and thus we know from [FI02] that the $\chi_{2\bar{2}}$-part of the intrinsic torsion of this corresponding almost hermitian structure vanishes. So the SU(3)-type must necessarily be $\chi_{1\bar{1}345}$. Additionally we have

Lemma 3.21. *Given an* SU(3) *manifold* (M^6, g, ϕ)*, a connection with skew torsion* $\tilde{\nabla}$ *is characteristic if and only if it preserves the spinor* ϕ*,*

$$\tilde{\nabla} = \nabla^c \iff \tilde{\nabla}\phi = 0.$$

Proof. Obvious, but just for the record: ∇^c is an SU(3)-connection, and $\mathrm{SU}(3) = \mathrm{Stab}(\phi)$ forces ϕ to be parallel. Conversely, if ϕ is $\tilde{\nabla}$-parallel, the connection must preserve any tensor defined in terms of the spinor, like ω and ψ_ϕ, cf. Lemma 3.3. To conclude one must recall that the characteristic connection is unique, see [AFH13]. □

To obtain the ultimative sufficient and necessary condition we need to impose an additional constraint on the torsion components χ_4, χ_5:

Theorem 3.22. *A Riemannian spin manifold* (M^6, g, ϕ) *admits a characteristic connection if and only if it has type* $\chi_{1\bar{1}345}$ *and* $\eta = \frac{1}{4}\delta\omega$*, where* $\delta = -{*}d{*}$ *is the co-derivative.*

Proof. Let ∇^c be the U(3)-characteristic connection, T its torsion. We shall determine in which cases $\nabla^c\phi = \nabla^c j(\phi) = 0$. First of all

$$0 = (\nabla^c_X \omega)(Y, Z) = -2(\nabla^c_X \phi, ZYj(\phi)) - 2g(Y, Z)(\nabla^c_X \phi, j(\phi)).$$

Thus we have $(\nabla^c_X \phi, ZYj(\phi)) = 0$ if $Y \perp Z$. But for all $Y \perp Z$ the spinors $YZj(\phi)$ span the space orthogonal to ϕ and $j(\phi)$. In conclusion, ∇^c is characteristic for the SU(3)-structure if and only if $(\nabla^c_X \phi, j(\phi)) = 0$. Now choose a local adapted basis $e_1, .., e_6$ with $J_\phi e_i = -e_{i+1}, i = 1, 3, 5$. Using the formula

$$\nabla^c_X \phi = \nabla_X \phi + \frac{1}{4}(X \lrcorner T)\phi$$

and $\omega(X, Y) = -(XY\phi, j(\phi))$ we arrive at $4\eta(X) = -(X \lrcorner T\phi, j(\phi)) = \omega(X \lrcorner T) = -T(X, \omega) = T(\omega, X) = -\sum_{i=1,3,5} T(e_i, J_\phi e_i, X)$. From $T(e_i, Je_i, X) = -T(Je_i, e_i, X) = T(e_{i+1}, Je_{i+1}, X)$ for $i = 1, 3, 5$, we infer

$$4\eta(X) = -\tfrac{1}{2}\textstyle\sum_{i=1}^{6} T(e_i, Je_i, X) = -\sum_{i=1}^{6}(\nabla_{e_i}\omega)(e_i, X) = \delta\omega(X)$$

because $0 = (\nabla^c_X \omega)(Y, Z) = (\nabla_X \omega)(Y, Z) - \frac{1}{2}(T(X, J_\phi Y, Z) + T(X, Y, J_\phi Z))$. □

Corollary 3.23. *If an* SU(3) *structure is of type*

- $\chi_1 \oplus \chi_3$ *there is a unique characteristic connection,*
- χ_2 *there is no characteristic connection,*
- $\chi_4 \oplus \chi_5$ *there is a unique characteristic connection if and only if* $\delta\omega = 4\eta$.

The next theorem will give an explicit formula for the torsion of ∇^c. It relies on the computation for the Nijenhuis tensor of Lemma 3.14.
Suppose M^6 has type $\chi_{1\bar{1}345}$, and decompose the intrinsic endomorphism into

$$S = \lambda J_\phi + \mu \mathrm{Id} + S_{34},$$

as explained in Notation 3.13.

Theorem 3.24. *Suppose* (M^6, g, ϕ) *has type* $\chi_{1\bar{1}345}$. *Then the characteristic torsion of the induced* U(3) *structure reads*

$$T(X,Y,Z) = 2\lambda\psi^J_\phi(X,Y,Z) - 2\mu\psi_\phi(X,Y,Z) - 2 \overset{XYZ}{\mathfrak{S}} \psi_\phi(S_{34}(X),Y,Z).$$

If $\eta = \frac{1}{4}\delta\omega$ *then* T *is the characteristic torsion of the* SU(3) *structure as well.*

Proof. With Lemma 3.15 we have

$$d\omega \circ J_\phi(X,Y,Z) \quad = \quad 6\lambda\psi^J_\phi(X,Y,Z) - 6\mu\psi_\phi(X,Y,Z) + 2 \overset{XYZ}{\mathfrak{S}} \psi_\phi(S_{34}(X),Y,Z)$$

The formula $T = N - d\omega \circ J$ (see [FI02]) together with Lemma 3.14 gives the result. □

Remark 3.25. Another way of proving Theorem 3.24 is given by the explicit formula of the intrinsic torsion in Proposition 3.4. For an arbitrary basis $e_1, .., e_6$ the torsion T is then given by $-\sum_i pr_{\mathfrak{su}(3)^\perp}(e_i \lrcorner T) \otimes e_i = 2\Gamma$, where $pr_{\mathfrak{su}(3)^\perp}$ denotes the projection $\mathfrak{so}(6) \to \mathfrak{su}(3)^\perp$ with $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^\perp$. This method is used for G_2 structures in Theorem 4.14. One considers the map

$$\begin{array}{ccccc} TM^* \otimes \mathfrak{m} & \xrightarrow{\quad\kappa\quad} & \Lambda^3(TM^*) & \xrightarrow{\quad\Theta\quad} & TM^* \otimes \mathfrak{m} \\ \psi^-(SX,Y,Z) + \frac{2}{3}\eta(X)\omega(Y,Z) & \longmapsto & \frac{1}{3} \overset{XYZ}{\mathfrak{S}} (\omega(SX,Y,Z) + \frac{2}{3}\eta(X)\omega(Y,Z)) & & \\ & & T & \longmapsto & \sum_i e_i \otimes (e_i \lrcorner T)_{\mathfrak{m}}. \end{array}$$

With a computer algebra system one shows for example $\Theta \circ \kappa_{|\chi_3} = \frac{1}{3}\mathrm{Id}_{\chi_3}$ and obviously we have $\Theta \circ \kappa_{|\chi_1} = \mathrm{Id}_{\chi_1}$. To conclude Theorem 3.24 one would also need to calculate the mixing 4 and the 5 parts, which thus is a little harder then the corresponding calculation in the G_2 case as in section 4.4.

Manifolds with parallel torsion are of particular interest. A motivation came from [Fr98]. By a Theorem of Kirichenko, nearly Kähler manifolds are of this type. Parallel torsion is used in [AF08] in dimensions 4 and 5, in [Fr07] in dimension 7 and in [Pu09] in dimension 8.
Suppose M^6 admits a characteristic connection ∇^c. If S is ∇^c-parallel, then T^c is parallel with respect to ∇^c. We now consider the torsion to be ∇^c-parallel. Since $\nabla^c\phi = 0$ with [ABBK13] or [FI02] an easy computation shows

Corollary 3.26. *Let* σ_T *be the* 4*-form given by* $\frac{1}{2}\sum_i(e_i \lrcorner T) \wedge (e_i \lrcorner T)$ *and let* Ric^c *be the Ricci curvature with respect to the connection* ∇^c. *Then*

$$(X \lrcorner \mathrm{Ric}^c) \cdot \phi = \frac{1}{2}(X \lrcorner dT) \cdot \phi = (X \lrcorner \sigma_T) \cdot \phi.$$

In particular in the case $\chi_{1\bar{1}}$ we have

$$\sigma_T = \lambda d(\psi_\phi^J) - \mu d\psi_\phi = 12(\lambda^2 + \mu^2) * \omega.$$

Example 3.27. We take the real 6-manifold $M^6 = \mathrm{SL}(2, \mathbb{C})$ and view it as the reductive space

$$\frac{\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(2)}{\mathrm{SU}(2)} = G/H$$

with diagonal embedding. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G and H, and set $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, so that

$$\mathfrak{m} = \{(A, B) \in \mathfrak{g} \mid A - \bar{A}^t = 0, tr(A) = 0, B + \bar{B}^t = 0, tr(B) = 0\}.$$

The almost complex structure

$$J(A, B) = (iA, iB)$$

defines a U(3) structure of type χ_3 (see [AFS05]).The characteristic connection $\nabla^c = \nabla + \frac{1}{2}T$ preserves a spinor ϕ, so ∇^c is also characteristic for the induced SU(3) structure, which is of type χ_{35}. But by Theorem 3.22 we have $\eta = 0$, so actually the SU(3) type is χ_3. But then ϕ is harmonic. More general it is easy to see that

Corollary 3.28. *For any spinor ϕ the condition $\phi \in \operatorname{Ker} D$ is equivalent to $T\phi = 0$ whenever ∇^c exists.*

Since for our example $T\phi = 0$ was shown in [AFS05] the equation $D\phi = 0$ may be employed if more convenient. From Theorems 3.9, 3.22 we also know, that if ∇^c exists and $D\phi = 0$ the SU(3) structure must be of type χ_3.
This example shows, that there are SU(3) structures of type different to $\chi_{1\bar{1}5}$ (in this case χ_3) with $\nabla^c T = 0$.

4 G_2 geometry

Let $(M, g,)$ be a 7-dimensional oriented Riemannian manifold. M is said to carry a G_2 structure if it admits a reduction to $G_2 \subset \mathrm{SO}(7)$; alternatively, this amounts to the choice of a generic 3-form Ψ. With respect to a local orthonormal frame $e_1, \ldots, e_7$, such a 3-form can locally be written as

$$\Psi = e_{123} + e_{145} + e_{167} + e_{246} - e_{147} - e_{347} - e_{356}.$$

Here and subsequently, we do not distinguish between vectors and covectors and denote the k-form $e_{i_1} \wedge .. \wedge e_{i_k}$ by $e_{i_1..i_k}$. G_2 manifolds were classified by Fernández and Gray in [FG82] into four classes $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.
Friedrich and Ivanov proved that there is a characteristic connection if and only if the structure is of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$; these manifolds are sometimes called *G_2 manifolds with torsion* or *G_2T manifolds* for short. In [FI02] a concrete description of the torsion can be found (we do not need the explicit formula here). We will often use the skew symmetric endomorphism $P(X, .)$ introduced in [FG82],

$$\Psi(X, Y, Z) = g(X, P(Y, Z)).$$

We will use this description mostly in Section 4 of Chapter II, since there we need a description of a G_2 structure without spinors. Since here we lift a G_2 structure to a Spin(7) structure in dimension 8, additionally the $(2, 1)$-tensor P we have $\bar{P}$, which is the $(3, 1)$-tensor, defining the Spin(7) structure. This makes the notation for the G_2 tensor P more convenient than the often used (for example in this thesis all through this section and Section 1 of Chapter II) notation

$$P(X, Y) = X \times Y.$$

In Section 1 of Chapter II we use the description via spinors, which again gives a new viewpoint on G_2 structures: As in Section 3 we have G_2 as a simply connected subgroup of $\mathrm{SO}(7)$ and are thus able to construct the lift $G_2 \subset \mathrm{Spin}(7)$, where again G_2 is the stabilizer of a spinor. To better understand this we take a closer look at the linear algebra in dimension 7.

4.1 Linear algebra in dimension 7

We consider $\mathbb{R}^7$ and the corresponding $\mathrm{Spin}(7)$ representation $\Delta_7 = \Delta_6$. As in the 6-dimensional case, we have a real structure α. Since α commutes with the Clifford multiplication, we can fix the following real 8-dimensional $\mathrm{Spin}(7)$ representation

$$\Delta := \{\phi \in \Delta_7 \mid \alpha(\phi) = \phi\}.$$

We denote the corresponding real scalar product by $(\, . \, , . \,)$. For any spinor $\phi \in \Delta$ and $X \in \mathbb{R}^7$ we have

$$(X \cdot \phi, \phi) = -(\phi, X \cdot \phi) = -(X \cdot \phi, \phi)$$

and thus $X \cdot \phi \perp \phi$ for all $X \in \mathbb{R}^7$. The space $\{X \cdot \phi \mid X \in \mathbb{R}^7\}$ is 7-dimensional and thus $\{X \cdot \phi \mid X \in \mathbb{R}^7\} = \phi^{\perp}$.
Given a spinor ϕ of length one, we can define a 3-form

$$\Psi_\phi(X, Y, Z) = (X \cdot Y \cdot Z \cdot \phi, \phi) \tag{I.6}$$

for $X, Y, Z \in \mathbb{R}^7$. This 3-form is generic and normalized: For a orthonormal basis $e_1, .., e_7$ of $\mathbb{R}^7$ for any $i, j = 1, .., 7$ with $i \neq j$ we have

$$\sum_k \Psi_\phi(e_i, e_j, e_k) = 1.$$

Such a 3-form has stabilizer G_2 in $\mathrm{SO}(7)$ and since G_2 is simple it can be lifted to $\mathrm{Spin}(7)$ and is also the stabilizer of the spinor ϕ.
Vice versa given a generic, normalized 3-form Ψ we can define a spinor ϕ of length one via equation (I.6). Note that this spinor is only defined up to a multiplication of ± 1. We get

Lemma 4.1. *There is a one to one correspondence of*

- *generic normalized* 3*-forms in* $\mathbb{R}^7$
- *reductions of* $\mathrm{SO}(7)$ *to* G_2
- *reductions of* $\mathrm{Spin}(7)$ *to* G_2
- *one dimensional subspaces in* Δ*.*

Thus the space of G_2 *structures on* $\mathbb{R}^7$ *is given by*

$$\mathbb{RP}(\Delta) = \mathbb{RP}(7) = \mathrm{SO}(7)/G_2.$$

Useful for further calculations is the following

Lemma 4.2. *We have*

$$\Psi_\phi \cdot \phi \,=\, 7\phi, \quad \Psi_\phi \cdot \phi^* \,=\, -\phi^* \quad \text{if} \;\; \phi^* \perp \phi\,, \quad (X \lrcorner \Psi_\phi) \cdot \phi = -3X \cdot \phi.$$

4.2 G_2 manifolds

Let (M^7, g, ϕ) always be a seven dimensional Riemannian manifold with Levi-Civita connection ∇ equipped with a global spinor ϕ of length one, inducing a G_2-structure given by a 3-form Ψ_ϕ. Due to the fact that $\{X\phi \mid X \in TM^7\} = \phi^\perp$ there exists an endomorphism $S : TM^7 \to TM^7$ satisfying

$$\nabla_X \phi = S(X)\phi.$$

We consider the corresponding 3-form $\Psi_\phi(X,Y,Z) := (X \cdot Y \cdot Z \cdot \phi, \phi)$ and the cross product

$$g(X \times Y, Z) = \Psi_\phi(X,Y,Z).$$

Lemma 4.3. *The cross product satisfies*

$$(X \times Y)\phi = -XY\phi - g(X,Y)\phi$$

Proof. We have

$$(YZ\phi, X\phi) = -(XYZ\phi, \phi) = -\Psi_\phi(X,Y,Z) = -g(X, Y \times Z) = -((Y \times Z)\phi, X\phi).$$

Thus we have $((X \times Y)\phi, \phi^*) = (-XY\phi - g(X,Y)\phi, \phi^*)$ for all $\phi^* \perp \phi$. For $\phi^* = \phi$ we have $((X \times Y)\phi, \phi^*) = 0$ and the equality is satisfied since $g(XY\phi, \phi) = -g(X,Y)(\phi,\phi)$. $\square$

We cite (see also [FG82] and the beginning of Section 4 in Chapter II)

Lemma 4.4.

$$*\Psi_\phi(V,W,X,Y) = \Psi_\phi(V,W,X \times Y) - g(V,X)g(W,Y) + g(V,Y)g(W,X).$$

and conclude

Lemma 4.5. *The map S is given by*

$$(\nabla_V \Psi_\phi)(X,Y,Z) = 2 * \Psi_\phi(S(V),X,Y,Z).$$

Proof. We calculate

$$\begin{aligned}
(\nabla_V \Psi_\phi)(X,Y,Z) &= (XZY\nabla_V\phi, \phi) + (XYZ\phi, \nabla_V\phi) \\
&= (XZYS(V)\phi, \phi) + (XYZ\phi, S(V)\phi) \\
&= (XZYS(V)\phi, \phi) - (S(V)XYZ\phi, \phi) \\
&= (XZYS(V)\phi, \phi) + (XS(V)YZ\phi, \phi) + 2g(S(V),X)(YZ\phi,\phi) \\
&= (XZYS(V)\phi, \phi) - (XYS(V)Z\phi, \phi) - 2g(S(V),Y)(XZ\phi,\phi) \\
&\quad +2g(S(V),X)(YZ\phi,\phi) \\
&= (XZYS(V)\phi, \phi) + (XYZS(V)\phi, \phi) + 2g(S(V),Z)(XY\phi,\phi) \\
&\quad -2g(S(V),Y)(XZ\phi,\phi) + 2g(S(V),X)(YZ\phi,\phi) \\
&= 2(XZYS(V)\phi,\phi) - 2g(S(V),Z)g(X,Y) + 2g(S(V),Y)g(X,Z) \\
&\quad -2g(S(V),X)g(Y,Z).
\end{aligned}$$

With Lemma 4.4 we get

$$\begin{aligned}
2 * \Psi_\phi(S(V),X,Y,Z) &= -2 * \Psi_\phi(X,Y,Z,S(V)) \\
&= -2[(XY(Z \times S(V))\phi, \phi) - g(X,Z)g(S(V),Y) \\
&\quad +g(X,S(V))g(Y,Z)] \\
&= 2(XYZS(V)\phi,\phi) + 2g(Z,S(V))(XY\phi,\phi) \\
&\quad +2g(X,Z)g(S(V),Y) - 2g(X,S(V))g(Y,Z) \\
&= 2(XYZS(V)\phi,\phi) - 2g(Z,S(V))g(X,Y) \\
&\quad +2g(X,Z)g(S(V),Y) - 2g(X,S(V))g(Y,Z).
\end{aligned}$$

$\square$

Proposition 4.6. *The intrinsic torsion of the G_2-structure $\Psi_\phi(X,Y,Z) = (XYZ\phi, \phi)$ is*

$$\Gamma = -\frac{2}{3} S \lrcorner \Psi_\phi$$

where $S \lrcorner \Psi_\phi(X,Y,Z) = \Psi_\phi(S(X),Y,Z)$.

Proof. As in the proof of Theorem 3.4 we have $\nabla_X\phi = \frac{1}{2}\Gamma(X)\phi$ and

$$\nabla_X\phi = S(X)\phi = -\frac{1}{3}(S(X)\lrcorner\omega)\phi.$$

For any $X \in TM$, the 2-form $X\lrcorner\omega$ is in $\mathfrak{g}_2^\perp$ with respect to the splitting $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp$. Thus $X \mapsto -\frac{1}{3}(S(X)\lrcorner)\omega$ is a $\mathfrak{g}_2^\perp$ valued 1-form. Again, the same argument as in the proof of Theorem 3.4 shows that this implies $\Gamma(X) = -\frac{2}{3}S(X)\lrcorner\omega$. $\square$

To classify G_2 structures we look at a decomposition of $\nabla\phi \in TM^*\otimes(\phi^\perp)$ due to the identification

$$TM^* \otimes TM \;\cong\; TM^* \otimes (\phi^\perp), \quad \eta \otimes X \mapsto \eta \otimes X \cdot \phi.$$

Then the splitting of $TM^* \otimes (\phi^\perp)$ is given by

$$\begin{aligned} TM^* \otimes (\phi^\perp) \;\; &= \;\; \{\sum_i e_i \otimes e_i \cdot \phi\} \\ &\oplus \;\; \{\sum_{ij} a_{ij} e_i \otimes e_j \cdot \phi \mid (a_{ij}) \in \mathfrak{g}_2\} \\ &\oplus \;\; \{\sum_{ij} a_{ij} e_i \otimes e_j \cdot \phi \mid (a_{ij}) \text{ is traceless symmetric }\} \\ &\oplus \;\; \{\sum_{ij} a_{ij} e_i \otimes e_j \cdot \phi \mid (a_{ij}) \in \mathbb{R}^7\}. \end{aligned}$$

and corresponds to the decomposition of endomorphisms of $\mathbb{R}^7$

$$\mathrm{End}(\mathbb{R}^7) = \mathbb{R}\cdot\mathrm{Id} \oplus S_0(\mathbb{R}^7) \oplus \mathfrak{g}_2 \oplus \mathbb{R}^7,$$

where $S_0(\mathbb{R}^7)$ denotes the symmetric, traceless endomorphisms of $\mathbb{R}^7$. Identifying the dimensions one can compare the classes to the classes given in [FG82].

Lemma 4.7. *We have the following correspondence of classes of G_2 structures*

class	*description*	*dimension*
$\mathcal{W}_1$	$S = \lambda \cdot \mathrm{Id}$	1
$\mathcal{W}_2$	$S \in \mathfrak{g}_2$	14
$\mathcal{W}_3$	$S \in S_0(\mathbb{R}^7)$	27
$\mathcal{W}_4$	$S \in \mathbb{R}^7 = \{V\lrcorner\Psi_\phi \mid V \in \mathbb{R}^7\}$	7

In particular, S is symmetric if and only if $S \in \mathcal{W}_1 \oplus \mathcal{W}_3$.

4.3 Spin formulation

Since $T^*M^7 \otimes TM^7 \cong T^*M^7 \otimes \phi^\perp$, $\eta \otimes X \mapsto \eta \otimes X\phi$ is an isomorphism we can read the known decomposition $T^*M^7 \otimes \phi^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ via ϕ.

Notation 4.8. As in the SU(3) case we will shorten for example $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ to $\mathcal{W}_{124}$.

The restricted Clifford product $m : T^*M^7 \otimes \phi^\perp \to \Delta$ splits the space $\mathcal{W}_{1234}$ as follows.

Theorem 4.9. *Let (M^7, g, ϕ) be a Riemannian G_2 manifold. Then ϕ is a harmonic spinor*

$$D\phi = 0$$

if and only if the underlying G_2-structure has type $\mathcal{W}_{23}$.

Proof. First, the spin representation splits as $\Delta = \mathbb{R}\phi \otimes \phi^\perp = \mathcal{W}_1 \oplus \mathcal{W}_4$, so we may write the intrinsic-torsion space as

$$TM^7 \otimes \phi^\perp = \Delta \oplus \mathcal{W}_{23}.$$

Yet the restricted Clifford product $m : T^*M^7 \otimes \phi^\perp \to \Delta$ is G_2-equivariant. Since $T^*M^7 \otimes (\phi^\perp) = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathbb{R}\phi \oplus \mathfrak{g}_2 \oplus S^2_0 T^*M^7 \oplus TM^7$ we have $\operatorname{Ker} m = \mathcal{W}_{23}$, and the assertion follows from the definition $D = m \circ \nabla$. □

Lemma 4.10. *The $\mathcal{W}_{24}$ part of the intrinsic torsion in terms of ϕ is given by*

$$\tfrac{1}{2}\delta\Psi_\phi(X,Y) = (X\phi, \nabla_Y\phi) - (Y\phi, \nabla_X\phi) + (D\phi, XY\phi) + g(X,Y)(D\phi,\phi).$$

Proof. To prove the claim we simply calculate, in some basis $(e_i)_{i=1..7}$,

$$\begin{aligned}
\delta\Psi_\phi(X,Y) &= -\sum(\nabla_{e_i}\Psi_\phi)(e_i,X,Y) = -\sum[(XYe_i\nabla_{e_i}\phi,\phi) + (XYe_i\phi,\nabla_{e_i}\phi)] \\
&= -2(XYD\phi,\phi) - \sum[-2g(e_i,Y)(X\phi,\nabla_{e_i}\phi) + 2g(e_i,X)(Y\phi,\nabla_{e_i}\phi) \\
&\quad +(e_iXY\phi,\nabla_{e_i}\phi)] \\
&= -2(XYD\phi,\phi) + 2(X\phi,\nabla_Y\phi) - 2(Y\phi,\nabla_X\phi) + (XY\phi, D\phi) \\
&= 2(D\phi, XY\phi) + 2(X\phi,\nabla_Y\phi) - 2(Y\phi,\nabla_X\phi) + 2g(X,Y)(D\phi,\phi).
\end{aligned}$$

□

Theorem 4.11. *The basic classes of G_2-manifolds are described by the behaviour of the spinor ϕ as follows*

class	*spinorial equation*
$\mathcal{W}_1$	$\nabla_X\phi = \lambda X\phi$ *with* $\lambda \in \mathbb{R}$
$\mathcal{W}_2$	$\nabla_{X\times Y}\phi = Y\nabla_X\phi - X\nabla_Y\phi + 2g(Y, S(X))\phi$
$\mathcal{W}_3$	$(X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi)$ *and* $\lambda = 0$
$\mathcal{W}_4$	$\exists V \in TM^7 : \nabla_X\phi = XV\phi + g(V, X)\phi$
$\mathcal{W}_{12}$	$\nabla_{X\times Y}\phi = -14\lambda[Y\nabla_X\phi - X\nabla_Y\phi + g(Y, S(X))\phi - g(X, S(Y))\phi]$
$\mathcal{W}_{13}$	$(X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi)$
$\mathcal{W}_{14}$	$\exists V, W \in TM^7 : \nabla_X\phi = XVW\phi - (XVW\phi, \phi)$
$\mathcal{W}_{23}$	$S\phi = 0$ *and* $\lambda = 0$, *or* $D\phi = 0$
$\mathcal{W}_{24}$	$(X\nabla_Y\phi, \phi) = -(Y\nabla_X\phi, \phi)$
$\mathcal{W}_{34}$	$3(X\phi, \nabla_Y\phi) - 3(Y\phi, \nabla_X\phi) = (S\phi, XY\phi)$ *and* $\lambda = 0$
$\mathcal{W}_{123}$	$(S\phi, X\phi) = 0$, *or* $D\phi = -7\lambda\phi$
$\mathcal{W}_{124}$	$(Y\nabla_X\phi, \phi) + (X\nabla_Y\phi, \phi) = -2\lambda g(X, Y)$
$\mathcal{W}_{134}$	$3(X\phi, \nabla_Y\phi) - 3(Y\phi, \nabla_X\phi) = (S\cdot\phi, XY\phi) - 7\lambda g(X, Y)$
$\mathcal{W}_{234}$	$\lambda = 0$

where $\lambda = -\frac{1}{7}(D\phi, \phi) : M \to \mathbb{R}$ *is a real function and* $X \times Y$ *denotes the cross product relative to* Ψ_ϕ.

Proof. For $\mathcal{W}_1$ there is actually nothing to prove, for the given equation is nothing but the Killing spinor equation characterising this type of manifolds [Fr80] (see also [BFGK91]).
If $S \in \mathcal{W}_2$ then S is skew-symmetric, which implies $S(X \times Y) = S(X) \times Y + X \times S(Y)$. Then

$$\begin{aligned}
\nabla_{X\times Y}\phi &= (-Y \times S(X) + X \times S(Y))\phi \\
&= YS(X)\phi + g(Y, S(X))\phi - XS(Y)\phi - g(X, S(Y))\phi \\
&= Y\nabla_X\phi - X\nabla_Y\phi + 2g(S(X), Y)\phi.
\end{aligned}$$

By taking the dot product with ϕ we re-obtain that S is skew symmetric.
For $\mathcal{W}_3$ we use Lemma 4.10:

$$\tfrac{1}{2}\delta\Psi_\phi(X, Y) = (D\phi, XY\phi) + g(X, Y)(D\phi, \phi) + (X\phi, \nabla_Y\phi) - (Y\phi, \nabla_X\phi).$$

This fact together with $\operatorname{Tr} S = -(D\phi, \phi)$ allows to conclude.
Suppose $S \in \mathcal{W}_4$. The vector representation $\mathbb{R}^7$ is $\{V \times . \mid V \in \mathbb{R}^7\}$, so if S is represented by V we have $\nabla_X\phi = V \times X = -VX\phi - g(V, X)\phi = XV\phi + g(V, X)\phi$. As for the rest, we shall only prove what is not obvious.
A structure is of type $\mathcal{W}_{23}$ if $(D\phi, \phi) = 0$ and $0 = \frac{1}{2}\sum_{i,j}\delta\Psi_\phi(e_i, e_j)\Psi_\phi(e_i, e_j, X)$ (see [FG82]). This is equivalent to

$$\begin{aligned}
0 &= \textstyle\sum_{i,j}[(D\phi, e_ie_j\phi) - 7g(e_i, e_j)\lambda + (e_i\phi, S(e_j)\phi) - (e_j\phi, S(e_i)\phi)](e_ie_jX\phi, \phi) \\
&= -\textstyle\sum_{i,j}(D\phi, e_ie_j\phi)(e_ie_j\phi, X\phi) + 2\sum_{i,j}(e_i\phi, S(e_j)\phi)(e_ie_jX\phi, \phi).
\end{aligned}$$

Using the fact that $\{e_ie_j\phi \mid i, j = 1..7\}$ is a basis of Δ we obtain $\sum_{i,j}(\phi^*, e_ie_j\phi)e_ie_j\phi = 6\phi_1 + (\phi^*, \phi)\phi$. Define

$$S\phi := \sum_{i,j} g(e_i, S(e_j))e_ie_j\phi$$

and get $0 = -6(D\phi, X\phi) - 2(S\phi, X\phi)$. Thus $3D\phi = -S\phi$ holds on $\phi^\perp$. If $\lambda = 0$ we then have

$$(S\phi, \phi) = \sum g(e_i, S(e_j))(e_ie_j\phi, \phi) = -\sum g(e_i, S(e_i)) = (D\phi, \phi) = 0.$$

A structure is of type $\mathcal{W}_{34}$ if $(D\phi, \phi) = 0$ and

$$3\delta\Psi_\phi(X, Y) = \tfrac{1}{2}\sum_{i,j}\delta\Psi_\phi(e_i, e_j)\Psi_\phi(e_i, e_j, X \times Y).$$

Due to the calculation above the right-hand side equals

$$\begin{aligned} -6(D\phi, (X \times Y)\phi) \quad &- \quad 2(S\phi, (X \times Y)\phi) \\ &= \quad 6(D\phi, XY\phi) - 42g(X, Y)\lambda + 2(S\phi, XY\phi) + 2g(X, Y)(S\phi, \phi). \end{aligned}$$

We have

$$3\delta\Psi_\phi(X, Y) = 6(D\phi, XY\phi) - 42g(X, Y)\lambda + 6(X\phi, \nabla_Y\phi) - 6(Y\phi, \nabla_X\phi)$$

thus the defining equation is equivalent to

$$3(X\phi, \nabla_Y\phi) - 3(Y\phi, \nabla_X\phi) = (S\phi, XY\phi) - 7g(X, Y)\lambda$$

and if $\lambda = 0$ we get $3(X\phi, \nabla_Y\phi) - 3(Y\phi, \nabla_X\phi) = (S\phi, XY\phi)$. As for $\mathcal{W}_{124}$, note that S satisfies

$$(Y\nabla_X\phi, \phi) + (X\nabla_Y\phi, \phi) = -2g(X, Y)\lambda$$

if it is skew. If symmetric, instead, it satisfies the equation if and only if $g(X, S(Y)) = g(X, Y)\lambda$, and thus if $S = \lambda\mathrm{Id}$. □

Remark 4.12. The spinorial equation for the class $\mathcal{W}_4$ defines a connection with vectorial torsion as mentioned in [AF06]. This connection is a G_2 connection since the defining spinor ϕ is parallel by definition.

4.4 Adapted connections

Let (M^7, g, ϕ) be a 7-dimensional spin manifold with Levi-Civita connection ∇. As usual we identify $(3, 0)$- and $(2, 1)$-tensors using g. As in section 3.4 we consider the space $\mathcal{A}^g$ of all metric connections and define the map

$$\Xi : End(TM) \to \mathcal{A}^g, \ S \mapsto \frac{2}{3}\Psi_\phi(S., ., .).$$

The prescription

$$\nabla^n := \nabla + \frac{2}{3}S\lrcorner\Psi_\phi$$

defines a canonical G_2-connection (it preserves ϕ), since $(X\lrcorner\Psi_\phi)\phi = -3X\phi$, $\Psi_\phi\phi = 7\phi$ and $\Psi_\phi\phi^* = -\phi^*, \forall\phi^* \perp \phi$(see Lemma 4.2):

$$\nabla^n_X\phi = \nabla_X\phi + \frac{1}{3}\Psi_\phi(S(X), ., .) \cdot \phi = S(X) \cdot \phi - S(X) \cdot \phi = 0.$$

The endomorphism S encodes the intrinsic torsion. Abiding by Cartan's formalism, the set of all metric connections is isomorphic to (compare description of $\mathbb{R}^7$, $\Lambda^3(\mathbb{R}^7)$ and $\mathcal{T}$ in Section 3.4)

$$\underbrace{\mathbb{R}^7}_{\Lambda^1\mathbb{R}^7} \oplus \underbrace{(\mathbb{R} \oplus \mathbb{R}^7 \oplus S^2_0\mathbb{R}^7)}_{\Lambda^3\mathbb{R}^7} \oplus \underbrace{(\mathfrak{g}_2 \oplus S^2_0\mathbb{R}^7 \oplus \mathbb{R}^{64})}_{\mathcal{T}}$$

under G_2, and this allows us to see

Proposition 4.13. *The four pure kinds of a G_2-manifold correspond to ∇^n living in:*

$$\mathcal{W}_1 \iff \nabla^n \in \Lambda^3 \qquad \mathcal{W}_2 \iff \nabla^n \in \mathcal{T}$$
$$\mathcal{W}_3 \iff \nabla^n \in \Lambda^3 \oplus \mathcal{T} \qquad \mathcal{W}_4 \iff \nabla^n \in \Lambda^1 \oplus \Lambda^3.$$

Proof. Since for $A \in G_2$ we have

$$A^{-1}SA \mapsto \Psi_\phi(A^{-1}SA.,.,.) = \Psi_\phi(AA^{-1}SA., A., A.) = \Psi_\phi(SA., A., A.)$$

and Ξ is G_2 equivariant. Comparing the dimensions of the corresponding modules, in the cases $\mathcal{W}_1$ and $\mathcal{W}_2$ the connection ∇^n must have skew symmetric torsion and cyclic traceless torsion. Algebraic computations show that for $S \in \mathcal{W}_3$ the $\Lambda^3(\mathbb{R}^7)$ part does not vanish,

$$0 \neq \frac{1}{3} \overset{X,Y,Z}{\mathfrak{S}} \Psi_\phi(SX, Y, Z) \in \Lambda^3(\mathbb{R}^7)$$

and neither does the $\mathcal{T}$ part

$$0 \neq \Psi_\phi(X, Y, Z) - \frac{1}{3} \overset{X,Y,Z}{\mathfrak{S}} \Psi_\phi(SX, Y, Z) \in \mathcal{T}.$$

For $S \in \mathcal{W}_4$ we have $S(X) = V \times X$ for some vector V and thus with Lemma 4.4 we get

$$\Psi_\phi(SX, Y, Z) = g(V \times X, Y \times Z) = g(V, Y)g(X, Z) - g(V, Z)g(X, Y) - *\Psi_\phi(V, X, Y, Z),$$

which is contained in $\mathbb{R}^7 \oplus \Lambda^3(\mathbb{R}^7)$. □

The connection of pure type $\mathcal{T}$ of the $\mathcal{W}_2$ case is discussed in greater detail in [CI07]. Now among all G_2-connections, ensuing from the proposition above, there exists a unique connection ∇^c with skew-symmetric torsion. Therefore we may write

$$S = \lambda \mathrm{Id} + S_3 + S_4 \in \mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4$$

with $S_3 \in S^2_0(TM^7)$ and $S_4 = V \lrcorner \Psi_\phi$ for some vector V.

Theorem 4.14. *Let (M^7, g, ϕ) be a Riemannian G_2 manifold of type $\mathcal{W}_{134}$. The characteristic torsion reads*

$$T^c(X, Y, Z) = -\frac{1}{3} \overset{XYZ}{\mathfrak{S}} \Psi_\phi((2\lambda \mathrm{Id} + 9S_3 + 3S_4)X, Y, Z).$$

Proof. Consider the projections

$$T^*M^7 \otimes \mathfrak{g}_2^\perp \xrightarrow{\kappa} \Lambda^3(T^*M^7) \xrightarrow{\Theta} T^*M^7 \otimes \mathfrak{g}_2^\perp$$
$$\Psi_\phi(SX, Y, Z) \to \frac{1}{3} \overset{XYZ}{\mathfrak{S}} \Psi_\phi(SX, Y, Z), T \to \quad \textstyle\sum_i e_i \otimes (e_i \lrcorner T)_{\mathfrak{g}_2^\perp}$$

A little computation shows that the composite $\Theta \circ \kappa$ is the identity map with eigenvalues $1, 0, 2/9, 2/3$ on the four respective summands $\mathcal{W}_i$. But from [FI02] we know that if $-2\Gamma = \Theta(T)$ for some 3-form T, then T is the characteristic torsion. □

5 Spin(7) structures

In a similar spirit as in the case of the other structures, an 8-dimensional oriented Riemannian manifold (M, g) is called a Spin(7) manifold if it has a reduction to Spin(7) ⊂ SO(8), and this is equivalent to the choice of a 4-form Φ which, in a local frame $e_1, \ldots, e_8$, can be written as

$$\Phi = \phi + *\phi, \text{ and } \phi = e_{1278} + e_{3478} + e_{5678} + e_{2468} - e_{2358} - e_{1458} - e_{1368}.$$

We define a skew symmetric endomorphism $P(X,Y,.)$ on TM via

$$g(P(X,Y,Z),V) \;=\; \Phi(X,Y,Z,V).$$

We extend the metric g to 3-forms on TM in the usual way, i. e. $g(W_1 \wedge W_2 \wedge W_3, V_1 \wedge V_2 \wedge V_3) = \det(g(W_i,V_i))$ for $V_i, W_j \in TM$. For 3-forms $\xi = \sum_{i<j<k} \xi_{ijk} e_{ijk}$ and $\eta = \sum_{i<j<k} \eta_{ijk} e_{ijk}$ let $\eta(\xi)$ be defined as

$$\eta(\xi) := \sum_{i<j<k} \xi_{ijk}\eta(e_i,e_j,e_k) = \sum_{i<j<k} \xi_{ijk}\eta_{ijk} = g(\eta,\xi).$$

We define $p(X)$ via

$$g(X,P(\xi)) \;=\; g(p(X),\xi)$$

for $X \in TM$ and a 3-form ξ on M ($P(\xi)$ is well defined, since P is totally skew symmetric). Spin(7) manifolds were classified by Fernández in [Fe86]: they split in the two classes $\mathcal{U}_1$ and $\mathcal{U}_2$. S. Ivanov proves in [Iv04] that such a manifold always carries a characteristic connection ∇^c. He also states, that there always is a ∇^c-invariant spinor ϕ, thus defining the structure. As done in Sections 3 and 4 for SU(3) and G_2 structures one could translate the defining relations for the classes of classification to spinorial equations. But note, that a spinor in dimension 8 does not always have Spin(7) as its stabilizer. So there is no correspondence of spinors to Spin(7) structures as we used them in dimensions 6 and 7.

Chapter II

Hypersurfaces, cones and generalized Killing spinors

In this chapter we want to focus on hypersurfaces $M \subset \bar{M}$. We always consider a G structure on M, which corresponds to a $\bar{G}$ structure on $\bar{M}$. We are able to calculate the corresponding classes. In the first section we look at 6 dimensional M where $G = \mathrm{SU}(3)$ and $\bar{G} = G_2$ on $\bar{M}$. Since in this case we have spinorial characterisations of both structures, we are able to do the calculations for a general hypersurface to then restrict ourself to the case of a twisted cone over an SU(3) manifold. For the two cases of almost contact structures and G_2 structures on M (corresponding to almost hermitian and Spin(7) structures on $\bar{M}$) we have to choose another technique, which we can only use on (twisted) cones. So in Section 2 we introduce this technique to use it in Sections 3 (the almost contact case) and 4 (the G_2 case).

In the back of any section we will discuss (generalized) Killing spinors with torsion. Therefore we consider a G structure with Levi-Civita connection ∇, such that there exists a characteristic connection with torsion T. We define the one-parameter family of metric connections

$$\nabla^s := \nabla + 2sT$$

and recall that ϕ^* is called a *generalized Killing spinor with torsion* (gKS) if

$$\nabla^s_X \phi^* = A(X) \cdot \phi^*$$

for some symmetric endomorphism A. In the case where $A = \lambda \mathrm{Id}$ is a multiple of the identity, this is the definition of a Killing spinor with torsion. The case $s = \frac{1}{4}$ corresponds to the characteristic connection; however, there are many geometric situations in which the Killing equation holds for values $s \neq 1/4$. Especially the equation for $s = \frac{n-1}{4(n-3)}$ giving the limiting case of the eigenvalue inequality for the Dirac operator with torsion (see [ABBK13]) is interesting.
If additionally we have $s = 0$ we know that for λ real this is the definition of a real Killing spinor and λ is constant. Such spinors realize the equality case of the inequality for the eigenvalue of the Dirac operator (see [Fr80]) and in dimensions 6 and 7 this spinors correspond to nearly Kähler structures and nearly parallel G_2 structures (see Chapter I).
If $s = 0$ and A is arbitrary symmetric, this is the equation of a so called generalized Killing spinor (see [BGM05] and [FK01]). We saw in Chapter I that such spinors give half flat and cocalibrated structures in dimensions 6 and 7. As the Weingarten map of a hypersurface $M \subset \bar{M}$ is symmetric, the hypersurface theory can be used to construct generalized Killing spinors with (and without, being the case where $s = 0$) torsion.

1 $\mathrm{SU}(3)$ hypersurfaces in G_2 manifolds

Let $(\bar{M}^7, \bar{g}, \phi)$ be a 7-dimensional G_2 manifold and M^6 a hypersurface with transverse unit direction V

$$T\bar{M}^7 = TM^6 \oplus \langle V \rangle \,. \tag{II.1}$$

By restriction the spin bundle $\bar{\Sigma}$ of $\bar{M}^7$ gives a Spin(6)-bundle Σ over M^6, and similarly the Clifford multiplication $\cdot$ of M^6 is $X \cdot \phi = VX\phi$ in terms of the one on $\bar{M}^7$ (whose symbol we suppress, as usual). In particular, this implies that any $\sigma \in \Lambda^{2k} M^6 \subset \Lambda^{2k} \bar{M}^7$ of even degree will satisfy $\sigma \cdot \phi = \sigma\phi$. This notation was also used in [BGM05] to describe almost Killing spinors (compare Section 1.2 in this chapter). The second fundamental form $g(W(X), Y)$ of the immersion (W is the Weingarten map) accounts for the difference between the two Riemannian structures, and in $\bar{\Sigma}$ we can compare

$$\bar{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} VW(X)\phi,$$

where ∇ and $\bar{\nabla}$ are the Levi-Civita connections of M^6 and $\bar{M}^7$ A global spinor ϕ on $\bar{M}^7$ (a G_2-structure) restricts to a spinor ϕ on M^6 (an SU(3)-structure). The next lemma explains how both the almost complex structure and the spin structure are – in practice – induced by ϕ and the normal V.

Lemma 1.1. *For any sections $\phi^* \in \Sigma$ and vectors $X \in TM^6$*

i) $V\phi^* = j(\phi^*)$

ii) $VX\phi = (J_\phi X)\phi$

Proof. The volume form σ_7 satisfies $\sigma_7 \phi^* = -\phi^*$ for any spinor $\phi^* \in \Sigma$. Therefore $Vj(X\phi) = \sigma_7(X\phi) = -X\phi$. □

Another way of interpreting the structure on M^6 is to say that the G_2-form Ψ_ϕ defines J by

$$V \lrcorner \Psi_\phi = -\omega.$$

Lemma 1.2. *With respect to decomposition* (II.1) *the G_2-endomorphism of $\bar{M}^7$ has the form*

$$\bar{S} = \begin{pmatrix} J_\phi S - \frac{1}{2} J_\phi W & * \\ \eta & ** \end{pmatrix} \tag{II.2}$$

where (S, η) are the intrinsic tensors of M^6, J_ϕ the almost complex structure, W the Weingarten map.

This result was first proved in [CS06] using Cartan-Kähler theory. Our alternative argument is much simpler:

Proof. From the definition $\nabla_X \phi = VS(X)\phi + \eta(X)V\phi$ and Lemma 1.1 we get $\bar{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} VW(X)\phi = J_\phi S(X)\phi - \frac{1}{2} J_\phi W(X)\phi + \eta(X)V\phi$. □

The starred terms in matrix (II.2) should point to the half-obvious fact that the derivative $\nabla_V \phi$ cannot be reconstructed from S and η. As a matter of fact, later we will need to know that the bottom row of $\bar{S}$ is controlled by the product $(\nabla\phi, V\phi)$, so that the entry $**$ vanishes when $\nabla_V \phi = 0$.

Now we are ready for the main theorems, which explain how to go from M^6 to $\bar{M}^7$ (Theorem 1.4) and backwards (Theorem 1.5). The run-up to those requires the following preparatory definition. Recall that the map S is symmetric if the SU(3) structure is of type $\chi_{\bar{1}\bar{2}3}$ (see Lemma 3.6, Chapter I).

Definition 1.3. The symmetry of the Weingarten endomorphism W expresses half-flatness, i.e. class $\chi_{\bar{1}\bar{2}3}$, by [CS06]. Motivated by that we shall say that a hypersurface $M^6 \subset \bar{M}^7$ has

(0) *type zero* if W is the trivial map (meaning $\bar{\nabla} = \nabla$),

(I) *type one* if W is of class $\chi_{\bar{1}}$,

(II) *type two* if W is of class $\chi_{\bar{2}}$,

(III) *type three* if W is of class χ_3.

Due to the freedom in choosing entries in (II.2), we will take the easiest option (probably also the most meaningful one, geometrically speaking) and consider embeddings where $\nabla_V \phi = 0$.

Theorem 1.4. *Embed* (M^6, g, ϕ) *in some* $(\bar{M}^7, \bar{g}, \phi)$ *as in* (II.1)*, and suppose the* G_2*-structure to be parallel in the normal direction:* $\bar{\nabla}_V \phi = 0$.
Then the classes $\mathcal{W}_\alpha$ *of* $(\bar{M}^7, \bar{g}, \phi)$ *depend on the column position (the class of* M^6*) and the row position (the Weingarten type of* M^6*) as in the table*

	χ_1^+	χ_1^-	χ_2^+	χ_2^-	χ_3	χ_4	χ_5
0	$\mathcal{W}_{13}$	$\mathcal{W}_4$	$\mathcal{W}_3$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_2$	$\mathcal{W}_{234}$
I	$\mathcal{W}_{134}$	$\mathcal{W}_4$	$\mathcal{W}_{34}$	$\mathcal{W}_{24}$	$\mathcal{W}_{34}$	$\mathcal{W}_{24}$	$\mathcal{W}_{234}$
II	$\mathcal{W}_{123}$	$\mathcal{W}_{24}$	$\mathcal{W}_{23}$	$\mathcal{W}_2$	$\mathcal{W}_{23}$	$\mathcal{W}_2$	$\mathcal{W}_{234}$
III	$\mathcal{W}_{13}$	$\mathcal{W}_{34}$	$\mathcal{W}_3$	$\mathcal{W}_{23}$	$\mathcal{W}_3$	$\mathcal{W}_{23}$	$\mathcal{W}_{234}$

Proof. Let A be an endomorphism of $\mathbb{R}^6$ and θ a covector. Then the endomorphism $\bar{A} = \begin{pmatrix} J_\phi A & 0 \\ \theta & 0 \end{pmatrix}$ of $\mathbb{R}^7$ is of type $\mathcal{W}_4$ if and only if $\theta = 0$ and A is a multiple of the identity, since J_ϕ is given by $g(X, J_\phi Y) = \frac{1}{2}\Psi_\phi(V, X, Y)$.
With other similar and easy implications we show that the type of $\bar{A} = \begin{pmatrix} J_\phi A & 0 \\ \theta & 0 \end{pmatrix}$ is determined by the class of the intrinsic tensors (A, θ) on M^6 in the following way:

$(A,\theta) \in$	χ_1	χ_1^-	χ_2^+	χ_2^-	χ_3	χ_4	χ_5
$(J_\phi A, \theta) \in$	$\chi_{\bar{1}}$	χ_1^+	χ_2^-	χ_2^+	χ_3	χ_4	χ_5
$\bar{A} \in$	$\mathcal{W}_{13}$	$\mathcal{W}_4$	$\mathcal{W}_3$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_2$	$\mathcal{W}_{234}$

Now the theorem can be proved:: Consider for example an SU(3) structure (S, η) of type χ_3 on a hypersurface of type one. Then $\begin{pmatrix} J_\phi S & 0 \\ \eta & 0 \end{pmatrix}$ is of type $\mathcal{W}_3$ and since W is a multiple of the identity, $\begin{pmatrix} J_\phi W & 0 \\ 0 & 0 \end{pmatrix}$ is of type $\mathcal{W}_4$. This immediately states that $\bar{S} = \begin{pmatrix} J_\phi S - \frac{1}{2} J_\phi W & 0 \\ \eta & 0 \end{pmatrix}$ and thus the type of the G_2 structure is $\mathcal{W}_{34}$. □

We will now do the opposite: start from the ambient space $(\bar{M}^7, \bar{g}, \phi)$ and infer the structure of its codimension-one submanifolds M^6. By inverting formula (II.2) we immediately see from

$$\bar{S}_{|TM^6} = JS + \frac{1}{2}JW \text{ implying } J_\phi \bar{S}_{|TM^6} = -S - \frac{1}{2}W$$

that

$$J_\phi \bar{S}_{|TM^6} + \frac{1}{2} W = -S$$

and thus

$$S = -J_\phi \bar{S}\big|_{TM^6} + \frac{1}{2} W, \qquad \eta(X) = g(\bar{S}X, V)$$

for any $X \in TM^6$.
To conclude, we can state the final result on hypersurfaces (which can be found, in a different form, in [C06], Section 4).

Theorem 1.5. *Let $(\bar{M}^7, \bar{g}, \phi)$ be a Riemannian spin manifold of class $\mathcal{W}_\alpha$. Then a hypersurface M^6 orthogonal to V for some $V \in T\bar{M}^7$ has an induced spin structure ϕ^+: its class is an entry in the matrix below that is determined by its column (the Weingarten type) and row position ($\mathcal{W}_\alpha$)*

	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_4$
0	χ_1	$\chi_{\overline{12}45}$	χ_{1235}	$\chi_{\bar{1}45}$
I	$\chi_{1\bar{1}}$	$\chi_{\overline{12}45}$	$\chi_{1\bar{1}235}$	$\chi_{\bar{1}45}$
II	$\chi_{1\bar{2}}$	$\chi_{\overline{12}45}$	$\chi_{12\bar{2}35}$	$\chi_{\overline{12}45}$
III	χ_{13}	$\chi_{\overline{12}345}$	χ_{1235}	$\chi_{\bar{1}345}$

Proof. To proceed as in Theorem 1.4, we prove that the class of an endomorphism $\bar{A} = \left(\begin{smallmatrix} J_\phi A & * \\ \theta & * \end{smallmatrix}\right)$ on $\mathbb{R}^7$ determines the class of (A, θ) on a $\mathbb{R}^6$ in the following way:

$\bar{A} \in$	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_4$
$(A, \theta) \in$	χ_1	$\chi_{\overline{12}45}$	χ_{1235}	$\chi_{\bar{1}45}$

If $\bar{A} \in \mathcal{W}_1$ we have $\bar{A} = \lambda \mathrm{Id}$ and thus $\eta = 0$ and $A = \lambda J_\phi$.
If $\bar{A} \in \mathcal{W}_2$ then $J_\phi A$ is skew-symmetric, and A has type $\chi_{\overline{12}4}$.
If $\bar{A}$ is of type $\mathcal{W}_3$ we have $\bar{S} = \left(\begin{smallmatrix} J_\phi A & \eta \\ \eta & -\mathrm{tr}(J_\phi A) \end{smallmatrix}\right)$ for some symmetric $J_\phi A$. Therefore JA is of type $\chi_{\overline{123}}$ implying the type χ_{123} for A.
Suppose $\bar{A} \in \mathcal{W}_4$, so there is a vector Z such that $g(X, \bar{A}Y) = \Psi_\phi(Z, X, Y)$, whence

$$(XYZ\phi, \phi) = (\bar{A}Y\phi, X\phi)$$

for every $X, Y \in \mathbb{R}^7$. Restrict this equation to $X, Y \in \mathbb{R}^6$ and put $Z = \lambda V + Z_1, Z_1 \in \mathbb{R}^6$. Then $J_\phi A = \lambda J_\phi + A_1$ with $(XYZ_1\phi, \phi) = (A_1 Y\phi, X\phi)$. Since A_1 is skew we have

$$\begin{aligned} g(X, A_1 J_\phi Y) &= (Z_1 X J_\phi Y\phi, \phi) = (Z_1 XVY\phi, \phi) = -(Z_1 YVX\phi, \phi) \\ &= -(Z_1 Y J_\phi X\phi, \phi) = -g(Y, A_1 J_\phi X) = -g(X, J_\phi A_1 Y), \end{aligned}$$

so $A_1 J_\phi = -J_\phi A_1$ and A_1 has type χ_4. Thus $J_\phi A \in \chi_{14}$. □

From this table it becomes clear that we cannot have a $\mathcal{W}_1$-manifold if the derivative of ϕ along V vanishes.
Moreover, in case $\nabla_V \phi = 0$ the χ_5-component disappears from everywhere, simplifying the matter a little.

1.1 Spin cones

As usual, start with (M^6, g, ϕ) with intrinsic torsion (S, η). Choose a complex-valued function $h = h_1 + ih_2 : I \to S^1$ of unit norm defined on some real interval I and set

$$\phi_t := h(t)\phi := h_1(t)\phi + h_2(t)j(\phi)$$

yielding a new (family of) SU(3) structures on M^6 depending on $t \in I$. Then $j(\phi)_t = j(\phi_t) = j(h_1(t)\phi + h_2(t)j(\phi)) = -h_2(t)\phi + h_1(t)j(\phi) = h(t)j(\phi)$. The product of a complex number $a \in \mathbb{C}$ with an endomorphism $A \in \mathrm{End}(TM)$ is defined as $aA = (\mathrm{Re}\, a)A + (\mathrm{Im}\, a)J_\phi A$. With this definitions we get $h(A(X)\phi^*) = (hA)(X)\phi^* = A(X)\bar{h}\phi^*$ for any spinor ϕ^*. The first observation is that the intrinsic torsion of (M^6, g, ϕ_t) is given by $(h^2 S, \eta)$ (cf. (ii) in Remark 3.18, with $f = h$ is constant on M^6), for

$$\begin{aligned}\nabla_X \phi_t &= h\nabla_X \phi = h(S(X)\cdot\phi) + h\eta(X)j(\phi)\\ &= (hS)(X)\cdot(\bar{h}h\phi) + \eta(X)j(\phi)_t = (h^2S)(X)\cdot\phi_t + \eta(X)j(\phi)_t.\end{aligned}$$

Now let us conformally rescale the metric by some positive function $f : I \to \mathbb{R}_+$ and consider

$$M^6_t := (M^6, f(t)^2 g, \phi_t).$$

Note that M^6 and M^6_t have one Levi-Civita connection and one spinor bundle Σ, but distinct Clifford multiplications $\cdot$, $\cdot_t$, albeit related by $X\cdot\phi^* = \frac{1}{f(t)} X \cdot_t \phi^*, \forall \phi^*$. Because $\nabla_X \phi_t = h^2 S(X)\cdot\phi_t + \eta(X)j(\phi)_t = \frac{h^2}{f} S(X)\cdot_t \phi_t + \eta(X)j(\phi)_t$ the intrinsic torsion of M^6_t gets rescaled as $(\frac{h^2}{f}S, \eta)$.

Definition 1.6. The metric cone

$$(\bar{M}^7, \bar{g}) = (M^6 \times I, f^2(t)g + dt^2)$$

equipped with spin structure $\bar{\phi} := \phi_t$ will be referred to as the *spin cone* over M^6.

The Levi-Civita connection $\bar{\nabla}^t$ of the cone reads

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{f'(t)}{f(t)}\bar{g}(X,Y)\partial_t$$

for $X, Y \in TM^6$, whence the Weingarten map is $W = -\frac{f'}{f}\mathrm{Id}$. Computing

$$\bar{\nabla}_{\partial_t}\bar{\phi} = \bar{\nabla}_{\partial_t} h\phi = h'\phi = -ih'j(\phi) = -i\frac{h'}{h}hV\phi = -i\frac{h'}{h}V\bar{\phi}$$

we get the intrinsic torsion of $\bar{M}^7$ by

$$\bar{S} = \begin{pmatrix} \frac{h^2}{f}J_\phi S + \frac{f'}{2f}J_\phi & 0 \\ \eta & -i\frac{h'}{h} \end{pmatrix}$$

By decomposing $S = \lambda J_\phi + \mu\mathrm{Id} + R \in \chi_1 \oplus \chi_{\bar{1}} \oplus \chi_{2345}$ the upper-left term can be written as

$$\frac{-\lambda \mathrm{Im}\, h^2 + \mu \mathrm{Re}\, h^2 + f'/2}{f} J_\phi + \frac{-\lambda \mathrm{Re}\, h^2 - \mu \mathrm{Im}\, h^2}{f}\mathrm{Id} + \frac{\mathrm{Re}\, h^2}{f}J_\phi R + \frac{-\mathrm{Im}\, h^2}{f}R.$$

Suppose we require $\bar{M}^7$ to be of type W_1: since $\bar{S}$ then is a multiple of the identity, we need h'/h to be constant, so $h(t) = \exp(i(ct + d))$, $c, d \in \mathbb{R}$. Let us see what happens for specific choices of structure on the hypersurface.

The sine cone. Start with an SU(3) manifold (M^6, g, ϕ) of type $\chi_{\bar{1}}$ with $S = -\frac{1}{2}\mathrm{Id}$. The choice $h = \mathrm{e}^{it/2}$ produces a cone

$$(\ M^6 \times (0,\pi), \ \ sin(t)^2 g + dt^2, \ \ e^{it/2}\phi \ ).$$

where $\bar{S} = \frac{1}{2}\mathrm{Id}$. This construction was considered in [FIVU08] and [St09], among others.

Other cones of pure type. To obtain different-type structures we start this time by fixing the function $h = 1$, so that $\bar{\phi} = \phi$ and

$$\bar{S} = \begin{pmatrix} \frac{\mu+\frac{1}{2}f'}{f} J_\phi - \frac{\lambda}{f}\mathrm{Id} + \frac{1}{f} J_\phi R & 0 \\ \eta & 0 \end{pmatrix}.$$

By selecting different SU(3) structures we have
a) Take M^6 to be $\chi_{\overline{1}\overline{2}4}$, say $S = \mu\mathrm{Id} + R$, and $\mu < 0$ constant: the cone

$$(\ M^6 \times \mathbb{R}_+, \ \ 4\mu^2 t^2 g + dt^2, \ \ \phi \ )$$

has $\bar{S} = \left(\begin{smallmatrix} -\frac{1}{2\mu t} J_\phi R & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and thus carries a calibrated G_2-structure.

b) On M^6 of type $\chi_{\bar{1}23}$ with $\mu < 0$ constant we can build the same cone, but now the resulting type will be $\mathcal{W}_3$.

c) Take a $\chi_{\bar{1}}$-manifold ($S = \mu\mathrm{Id}$). Since $\left(\begin{smallmatrix} k(t)J_\phi & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is of type $\mathcal{W}_4$ irrespective of the map $k(t)$, the cone

$$(\ M^6 \times I, \ \ f(t)^2 g + dt^2, \ \ \phi \ )$$

is always $\mathcal{W}_4$ if we start from $\chi_{\bar{1}}$, since R and λ vanish in this case. When $\mu < 0$ the special choice $f(t) = -2\mu t$ will additionally give $\bar{S} = 0$. This Ansatz is used in [Bä93] to describe real Killing spinors on Riemannian manifold M^6.

1.2 Killing spinors with torsion

Let $(\bar{M}^7, \bar{g}, \phi)$ be a G_2 manifold with characteristic connection $\bar{\nabla}^c$ and torsion $\bar{T}$, and suppose (M^6, g, ϕ) is a submanifold of type one or three, such that $V \lrcorner \bar{T} = 0$, cf. (II.1). The latter equation warrens that $\bar{T}$ can be restricted to a 3-form on M^6. We decompose the Weingarten map $W = \mu\mathrm{Id} + W_3$ with $JW_3 = -W_3 J$ and prove

Lemma 1.7. *The differential form*

$$L(X,Y,Z) := -\overset{XYZ}{\mathfrak{S}} \ \psi_\phi(W_3(X), Y, Z) - \mu\psi_\phi(X,Y,Z)$$

satisfies $(X \lrcorner L)\phi = -2W(X)\phi$.

Proof. In an arbitrary basis $e_1, .., e_6$ the torsion is $-\sum_i (e_i \lrcorner T)_{\mathfrak{su}(3)^\perp} \otimes e_i = 2\Gamma$, where $(e_i \lrcorner T)_{\mathfrak{su}(3)^\perp}$ denotes the projection of $e_i \lrcorner T$ by $\mathfrak{so}(6) \to \mathfrak{su}(3)^\perp$. It is not hard to see that

$$\begin{aligned} &T^* M^6 \otimes \mathfrak{su}(3)^\perp \xrightarrow{\kappa} \Lambda^3(T^*M^6) \xrightarrow{\Theta} T^*M^6 \otimes \mathfrak{su}(3)^\perp \\ &S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \omega \mapsto \frac{1}{3}\ \mathfrak{S}\ (S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \omega); T \mapsto \sum_i e_i \otimes (e_i \lrcorner T)_{\mathfrak{su}(3)^\perp} \end{aligned}$$

satisfy $\Theta \circ \kappa_{|\chi_3} = \frac{1}{3}\mathrm{Id}_{\chi_3}$ and $\Theta \circ \kappa_{|\chi_1} = \mathrm{Id}_{\chi_1}$.
But since SU(3) is the stabilizer of ϕ for any 3-form $R \in \Lambda^3(T^*M^6)$ we have $R(X) \cdot \phi = \Theta(R)(X) \cdot \phi$, so

$$\begin{aligned}(X \lrcorner L) \cdot \phi &= X \lrcorner (L) \cdot \phi = X \lrcorner (\Theta \circ \kappa(-\psi_\phi(3W_3 + \mu \mathrm{Id}., ., .))) \cdot \phi = X \lrcorner (-\psi_\phi(W_3 + \mu \mathrm{Id}., ., .)) \cdot \phi \\ &= -\psi_\phi(W(X), ., .) \cdot \phi = -2W(X) \cdot \phi,\end{aligned}$$

proving the lemma. □

For $X \in TM^6$ we have

$$\begin{aligned}0 &= \bar\nabla^c_X \phi = \bar\nabla_X \phi + \tfrac{1}{4}(X \lrcorner \bar T)\phi \\ &= \nabla_X \phi + \tfrac{1}{4}(X \lrcorner \bar T) \cdot \phi - \tfrac{1}{2} W(X) \cdot \phi.\end{aligned}$$

So if we define $T := \bar T_{|M^6} + L$ then $\nabla^c := \nabla + T$ is characteristic on (M^6, g, ϕ). Hence if there exist characteristic connections on $\bar M^7$ and M^6, their difference must be L.
Note that the restriction $\bar\nabla_V \phi = 0$ on $\bar M^7$ is not strong enough, since this only implies $(V \lrcorner \bar T)\phi = 0$ and not $V \lrcorner \bar T = 0$.

In the beginning of this chapter we introduced generalized Killing spinor (gKS) ϕ^*

$$\nabla^s \phi^* = A(X) \cdot \phi^*$$

for some symmetric endomorphism $A : TM^6 \to TM^6$, where $\nabla^s := \nabla + 2sT$. Starting with any G_2 manifold of type $\mathcal{W}_{13}$, the corresponding spinor is a generalized Killing spinor (without torsion), since it satisfies $\nabla_X \phi = S(X)\phi$ with symmetric S (see Lemma 4.7 in Chapter I). Examples may arise from 3-Sasaki manifolds as discussed in [AF10].
Suppose the restriction to M^6 of a spinor ϕ^* is a gKS. Then at any point of M^6

$$\begin{aligned}\bar\nabla^s_X \phi^* &= \bar\nabla_X \phi^* + s(X \lrcorner \bar T)\phi^* = \nabla_X \phi^* + s(X \lrcorner \bar T)\phi^* - \tfrac{1}{2} V W(X)\phi^* \\ &= \nabla^s_X \phi^* + s(X \lrcorner (\bar T - T))\phi^* - \tfrac{1}{2} V W(X)\phi^* \\ &= V(A - \tfrac{1}{2}W)(X)\phi^* + s(X \lrcorner L)\phi^*.\end{aligned}$$

Choosing $A = \frac{1}{2}W$ annihilates one term so that $\bar\nabla^s_X \phi^* = s(X \lrcorner L)\phi^*$.
Conversely, any $\bar\nabla^s$-parallel spinor on $\bar M^7$ satisfies

$$0 = \bar\nabla^s_X \phi^* = \nabla^s_X \phi^* + s(X \lrcorner (\bar T - T)) \cdot \phi^* - \tfrac{1}{2} W(X) \cdot \phi^*.$$

To summarise,

Theorem 1.8. *Let $(\bar M^7, \bar g, \phi)$ be a G_2 manifold with characteristic connection $\bar\nabla^c$ and torsion $\bar T$. Let $M^6 \subset \bar M^7$ be a hypersurface, of type one or three, such that $V \lrcorner \bar T = 0$. Then $(M^6, g = \bar g_{|TM^6}, \phi)$ is an* SU(3) *manifold and*
i) the characteristic connection of M^6 is $\nabla + \bar T + L$.
ii) Any solution ϕ^* on $\bar M^7$ to the gKS equation $\nabla^s_X \phi^* = \frac{1}{2} W(X) \cdot \phi^*$ on M^6 must satisfy

$$\bar\nabla^s_X \phi^* = s(X \lrcorner L)\phi^*.$$

iii) Vice versa, if ϕ^* is $\bar\nabla^s$-parallel on $\bar M^7$ then it solves

$$\nabla^s_X \phi^* = -sX \lrcorner (\bar T - T) \cdot \phi^* + \frac{1}{2} W(X) \cdot \phi^*.$$

Example 1.9. Given (M^6, g) we build the twisted cone

$$(\bar{M}^7 := M^6 \times \mathbb{R}, \bar{g} := a^2t^2 g + dt^2)$$

for some $a > 0$. From the submanifold $M^6 \cong M^6 \times \{\frac{1}{a}\} \subset \bar{M}^7$ we can only compute the Clifford multiplication of $\bar{M}^7$ in the points of $M^6 \times \{\frac{1}{a}\}$. As in section 1.1 we therefore consider $M_t^6 := (M^6, a^2t^2g) \cong M_t^6 \times \{\frac{t}{a}\} \subset \bar{M}^7$ as a hypersurface. At any point in M_t^6 the spinor bundle of M_t^6 and of $\bar{M}^7$ are the same and they can be identified with the spinor bundle of M^6. We get $X \cdot \phi^* = \frac{1}{at}\partial_t X\phi^*$. Since the metric of M_t^6 is just a scaling of the metric of M^6, their Levi-Civita connections are the same. For the Levi-Civita connection ∇ on M^6 and the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}^7$ we have

$$\bar{\nabla}_X\phi^* = \nabla_X\phi^* + \frac{1}{2t}\partial_t X\phi^* = \nabla_X\phi^* + \frac{a}{2}X \cdot \phi^*,$$

as $W(X) = -\frac{1}{t}X$. Therefore the submanifolds M_t^6 are of type one, and one could determine the possible structures using Theorems 1.4, 1.5. Any 2-form σ on M^6 is a 2-form on $\bar{M}^7$ with $\partial_t \lrcorner \sigma = 0$, and in addition

$$\sigma \cdot \phi^* = a^2t^2\sigma\phi^*$$

for any spinor ϕ^*.

Let ϕ be an SU(3)-structure on M^6. We consider the G_2-structure on $\bar{M}^7$ given by ϕ. Then $\partial_t \lrcorner \Psi_\phi = -a^2t^2\omega$. If M^6 has a characteristic connection ∇^c with torsion T we have

$$\begin{aligned} 0 &= \nabla_X^c\phi = \nabla_X\phi + \tfrac{1}{4}(X \lrcorner T)\cdot\phi = \bar{\nabla}_X\phi - \tfrac{a}{2}X\cdot\phi + \tfrac{1}{4}(X\lrcorner T)\cdot\phi \\ &= \bar{\nabla}_X\phi - \tfrac{a}{4}(X\lrcorner\psi_\phi)\cdot\phi + \tfrac{1}{4}(X\lrcorner T)\cdot\phi = \bar{\nabla}_X\phi + \tfrac{1}{4}(X\lrcorner(T - a\psi_\phi))\cdot\phi \\ &= \bar{\nabla}_X\phi + \tfrac{1}{4}(X\lrcorner a^2t^2(T - a\psi_\phi))\phi, \end{aligned}$$

showing that $\bar{T} = a^2t^2(T - a\psi_\phi)$ is the characteristic torsion of $\bar{M}^7$.

Given a $\bar{\nabla}^s$-parallel spinor ϕ^*

$$\begin{aligned} 0 &= \bar{\nabla}_X\phi^* + s(X\lrcorner\bar{T})\phi^* = \nabla_X\phi^* + \tfrac{a}{2}X\cdot\phi^* + \tfrac{s}{a^2t^2}(X\lrcorner\bar{T})\cdot\phi^* \\ &= \nabla_X\phi^* + \tfrac{a}{2}X\cdot\phi^* + s(X\lrcorner(T - a\psi_\phi))\cdot\phi^* = \nabla_X^s\phi^* + \tfrac{a}{2}X\cdot\phi^* - as(X\lrcorner\psi_\phi)\cdot\phi^*, \end{aligned}$$

from which

$$\nabla_X^s\phi^* - as(X\lrcorner\psi_\phi)\cdot\phi^* = -\tfrac{a}{2}X\cdot\phi^*.$$

Consider the differential form $\bar{\psi_\phi} \in \Lambda^3\bar{M}^7$ defined by

$$\bar{\psi_\phi}(X, Y, Z) := a^3t^3\psi_\phi(X, Y, Z) \text{ for } X, Y, Z \in TM^6 \text{ and } \partial_t\lrcorner\bar{\psi_\phi} = 0$$

For a Killing spinor solving $\nabla_X^s\phi^* = -\frac{a}{2}X\cdot\phi^*$ we then have

$$\begin{aligned} 0 &= \nabla_X^s\phi^* + \tfrac{a}{2}X\cdot\phi^* = \nabla_X\phi^* + \tfrac{a}{2}X\cdot\phi^* + s(X\lrcorner T)\cdot\phi^* \\ &= \bar{\nabla}_X\phi^* + sa^2t^2(X\lrcorner T)\phi^* = \bar{\nabla}_X\phi^* + sa^3t^2(X\lrcorner\psi_\phi)\phi^* + s(X\lrcorner\bar{T})\phi^*. \end{aligned}$$

Consequently

$$0 = \bar{\nabla}_X^s\phi^* + \frac{s}{t}(X\lrcorner\bar{\psi_\phi})\phi^*.$$

Example 1.10. Let $(M^7, g, \xi, \eta, \psi)$ be a Einstein Sasaki manifold with Killing vector ξ, its dual η and endomorphism ψ such that $\psi^2 = -id$ on $\xi^\perp$. Let

$$g_t := tg + (t^2 - t)\eta\otimes\eta, \quad \xi_t := \frac{1}{t}\xi \quad \text{and} \quad \eta_t := t\eta$$

for $t > 0$ (compare also Example 3.18). Becker-Bender proved in Lemm 2.18 of [Be12] that $(M^7, g_t, \xi_t, \eta_t, \psi)$ again is Sasaki. Let furthermore ∇^{g_t} be the Levi-Civita connection on $(M, g_t, \xi_t, \eta_t, \psi)$ and T^{g_t} the characteristic torsion of the almost contact structure (there exists a characteristic connection since the manifold is Sasaki). Then Becker-Bender proved in Theorem 2.22 of [Be12] the existence of a Killing spinor with torsion for the connection

$$\nabla^{g_t}_X + (\frac{1}{2t} - \frac{1}{2})(X \lrcorner T^{g_t}).$$

The introduction of so called *quasi Killing spinors* in [FK00] is a more restricted version of our definition of a generalized Killing spinor and leads to a gKS on the Sasaki manifold $(M^7, g_t, \xi_t, \eta_t, \psi)$. As stated in [Be12], in this example the gKS and the Killig spinors with torsion are the same. As for a gKS we have

$$\nabla_X \phi^* = A(X) \cdot \phi^*$$

for some A symmetric, the G_2 structure given by this spinor is cocalibrated (of type $\mathcal{W}_{13}$).

Remark 1.11. The sign of the Killing constant may be reversed by $\phi^* \rightsquigarrow j(\phi^*)$.
The investigation of generalized Killing spinors (with and without torsion) can be pursued in many other contexts. To name but one, the five-dimensional picture was studied by Conti and Salamon [CS07].

2 Connections on cones and the cone construction on spinors

In Section 1 we considered a 6-dimensional hypersurface in a 7-dimensional manifold. As already mentioned in the beginning of this chapter, fortunately in this case, every spinor has the same stabilizer, giving us a correspondence of SU(3) (resp. G_2) structures and spinors. This lightens the calculation of the correspondences of SU(3) and G_2 structures on hypersurfaces. Unfortunately in dimensions higher then 7 the stabilizer of a spinor depends on the spinor. Thus in the more general case of an almost contact structure in dimension $2n + 1$ as a hypersurface of an almost hermitian manifold, we are not able to use the technique introduced in Section 1. The same holds for a G_2 hypersurface of a Spin(7) manifold.

Therefore we introduce a new method for these cases. We restrict ourselves to the case of a twisted cone over a manifold and use a technique of different connections to compare the structures given on a manifold and its cone. This technique was used earlier in several cases, see for example [Bä93].

2.1 The cone construction

Consider a Riemannian spin manifold (M, g) equipped with a metric connection ∇ with skew symmetric torsion T and connection form ω, meaning that the tensor $g(T(X, Y), Z) = g(\nabla_X Y - \nabla_Y X - [X, Y], Z)$ is skew symmetric. The aim of this Section is to generalize Bär's cone construction [Bä93] for Riemannian Killing spinors, i. e. the case when $\nabla = \nabla^g$. As an intermediate tool, we define a connection $\tilde{\nabla}$ on the spinor bundle by

$$\tilde{\nabla}_X \phi = \nabla_X \phi + \alpha X \cdot \phi, \text{ with } \alpha \in \mathbb{R}\backslash\{0\}.$$

Denote by $\mathcal{C}(\mathbb{R}^n)$ the Clifford algebra of $\mathbb{R}^n$ with respect to the standard negative definite euclidean scalar product, and by Δ_n the spin module of Spin(n). We consider the Clifford multiplication for $X \in \mathbb{R}^n \subset \mathcal{C}(\mathbb{R}^n)$ in Δ_n. It is the action of an element of $\mathbb{R}^n \subset \mathfrak{spin}(n) \oplus \mathbb{R}^n = \mathfrak{spin}(n+1) \subset \mathcal{C}(\mathbb{R}^n)$ in Δ_n. Let $P_{\mathrm{SO}(n)}M$ be the SO(n)-principal bundle of frames, ΣM the spinor bundle and $\rho_n : \mathcal{C}(n) \to GL(\Delta_n)$ the representation of the Clifford algebra, i. e. $\rho_{*|\mathfrak{spin}(n)}$ is the $\mathfrak{spin}(n)$ representation. Let $P_{\mathrm{Spin}(n)}M$ be the Spin(n)-principal bundle. For a local section

h in $P_{\mathrm{SO}(n)}M$, we identify TM and $P_{\mathrm{SO}(n)}M \times_{SO(n)} \mathbb{R}^n$ via $X = [h, \eta(dh(X))]$, where η is the solder form. The affine connection $\tilde{\nabla}$ induces a connection in the $\mathrm{Spin}(n+1)$-principal bundle $P_{\mathrm{Spin}(n)}M \times_{\mathrm{Spin}(n)} \mathrm{Spin}(n+1)$ as follows. Let

$$\Phi: \ P_{\mathrm{Spin}(n)}M \to P_{\mathrm{SO}(n)}, \quad \theta: \ \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

be the usual projections. We look at $\mathfrak{spin}(n+1) \cong \mathfrak{spin}(n) \oplus \mathbb{R}^n \subset \mathcal{C}(n)$, the restriction of ρ_* to $\mathfrak{spin}(n+1)$, and obtain for a local section k in $P_{\mathrm{Spin}(n)}M$ with $\Phi(k) = h$ and $\Sigma M \ni \phi = [k, \sigma]$,

$$\begin{aligned} \tilde{\nabla}_X[k,\sigma] &= \nabla_X[k,\sigma] + \alpha \cdot [h, \eta(dhX)] \cdot [k,\sigma] \\ &= [k, d\sigma(X) + \rho_*(\theta_*^{-1}(\omega(dhX)) + \alpha\eta(dhX)))\sigma]. \end{aligned}$$

Thus we get the $\mathfrak{spin}(n+1)$-valued 1-form $\hat{\omega} := \Phi^*(\theta_*^{-1}\omega + \alpha\eta)$ on $P_{\mathrm{Spin}(n)}M$. We extend $\hat{\omega}$ to $P_{\mathrm{Spin}(n+1)}M$ as follows: For $b \in P_{\mathrm{Spin}(n)}M$ we have $T_bP_{\mathrm{Spin}(n+1)}M = T_bP_{\mathrm{Spin}(n)}M \oplus dL_b(\mathbb{R}^n)$, where $L_b : \mathrm{Spin}(n+1) \to P_{\mathrm{Spin}(n+1)}M, \ g \mapsto b \cdot g$ and define

$$\hat{\omega}(dL_bY) := Y \in \mathbb{R}^n \subset \mathfrak{spin}(n+1).$$

For any $b \in P_{\mathrm{Spin}(n)}M$ we further extend $\hat{\omega}$ in a $\mathrm{Spin}(n+1)$ equivariant way. One checks that the given form is a connection form. It is the connection form of the connection given by $\tilde{\nabla}$. As in [Bä93], we consider the $\mathrm{SO}(n+1)$-principal bundle

$$P_{\mathrm{SO}(n+1)}M \ := \ P_{\mathrm{SO}(n)}M \times_{\mathrm{SO}(n)} \mathrm{SO}(n+1)$$

and calculate the corresponding connection form $\tilde{\omega}$ given by $\theta_*^{-1}\Phi^*\tilde{\omega} = \hat{\omega}$ for the projections $\Phi : P_{\mathrm{Spin}(n+1)}M \to P_{\mathrm{SO}(n+1)}M$ and $\theta : \mathrm{Spin}(n+1) \to \mathrm{SO}(n+1)$ and get

$$\tilde{\omega} = \begin{bmatrix} \omega & -2\alpha\eta \\ 2\alpha\eta^t & 0 \end{bmatrix}.$$

We now consider the cone $(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2r^2g + dr^2)$ for some fixed $a > 0$ with principal $\mathrm{SO}(n)$-bundle of frames $P_{\mathrm{SO}(n+1)}\bar{M}$, Levi-Civita connection $\bar{\nabla}^{\bar{g}}$ with connection form $\bar{\omega}^{\bar{g}}$ and projection $\pi : \bar{M} \to M$. For simplicity, we will write $X \in TM$ for a lift to $\bar{M}$ of a vector field on M. We define a tensor $\bar{T}$ on $\bar{M}$ from the torsion tensor T of ∇ via

$$\bar{T}(X,Y) \ := \ T(X,Y) \ \text{ for } X, Y \perp \partial_r, \quad \partial_r \lrcorner \bar{T} \ = \ 0.$$

We will notationally not distinguish between the $(2,1)$ torsion tensors and the corresponding skew symmetric $(3,0)$-tensors obtained via $g(T(X,Y),Z)$. For the metrics $g, \bar{g}$ on M and $\bar{M}$, we have $a^2r^2T(X,Y,Z) = \bar{T}(X,Y,Z)$ for $X,Y,Z \perp \partial_r$. From $\bar{T}$, we define on $\bar{M}$ the connection

$$\bar{\nabla} \ := \ \nabla^{\bar{g}} + \frac{1}{2}\bar{T},$$

whose connection form is $\bar{\omega}$. For $p \in M$ and $s \in \mathbb{R}^+$, the tangent bundle of $\bar{M}$ splits into $T_{(p,s)}\bar{M} = T_pM \oplus \mathbb{R}$, where $d\pi(T\bar{M}) = TM$. Thus, for $X \in TM \subset T\bar{M}$, we will write "$X$" instead of "$d\pi X$". With a local orthonormal frame $(X_1, \ldots, X_n)$ of M we have an isomorphism of the last two vector bundles given by $(Y \in \mathbb{R}^{n+1})$

$$\psi: \ \pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M) \times_{\mathrm{SO}(n+1)} \mathbb{R}^{n+1} \to T\bar{M}, \ \ [(X_1,..,X_n,\partial_r),Y] \mapsto [(\frac{1}{ar}X_1,..,\frac{1}{ar}X_n,\partial_r),Y].$$

Thus we can view the connection $\bar{\omega}$ as a connection of $\pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)$, which we again call $\bar{\omega}$. We summarize the different principal bundles with corresponding connections and vector bundles in the following table:

bundle	connection form	vector bundle	manifold
$P_{\mathrm{SO}(n)}M$	ω	TM	M
$\tilde{P}_{\mathrm{SO}(n+1)}M$	$\tilde{\omega}$		M
$\pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)$	$\pi^*\tilde{\omega}$	$\pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)\times_{\mathrm{SO}(n+1)}\mathbb{R}^{n+1}$	$\bar{M}$
$P_{\mathrm{SO}(n+1)}\bar{M}$	$\bar{\omega}$	$T\bar{M}$	$\bar{M}$

To determine $\bar{\omega}$ for a local frame $h := (X_1,..,X_n,\partial_r)$ in $\pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)$, $X \in T\bar{M}$, we need to compute ($Y \in \pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)\times_{\mathrm{SO}(n+1)}\mathbb{R}^{n+1}$)

$$\psi^{-1}(\bar{\nabla}_X\psi(Y)) \;=\; [h, d(\eta(dhY))(X) + \bar{\omega}(dhX)\eta(dhY)].$$

Let $\tilde{h} := (\frac{1}{ar}X_1,..,\frac{1}{ar}X_n,\partial_r)$ be a local frame in $P_{\mathrm{SO}(n+1)}$. For $Y \in TM \subset \pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)\times_{\mathrm{SO}(n+1)}\mathbb{R}^{n+1}$ we locally have $Y = [h,(Y_1,..,Y_n,0)^t]$ for functions $Y_i : M \to \mathbb{R}$ and thus $\psi(Y) = [\tilde{h},(Y_1,..,Y_n,0)^t]$. Therefore $ar\psi(Y)$ is independent of r and thus a lift of a vector field on M. Using the O'Neill formulas [O'N83, p. 206], we compute for lifts X, Y of vector fields in TM and the Levi-Civita connection $\bar{\nabla}^{\bar{g}}$ of $\bar{M}$

$$\bar{\nabla}^{\bar{g}}_{\partial_r}\partial_r = 0, \quad \bar{\nabla}^{\bar{g}}_{\partial_r}X = \bar{\nabla}^{\bar{g}}_X\partial_r = \frac{1}{r}X, \quad \bar{\nabla}^{\bar{g}}_X Y = \nabla^g_X Y - \frac{1}{r}\bar{g}(X,Y)\partial_r.$$

Adding the torsion tensor $\bar{T}$, this implies

$$\bar{\nabla}_{\partial_r}\partial_r = 0, \quad \bar{\nabla}_{\partial_r}X = \bar{\nabla}_X\partial_r = \frac{1}{r}X, \quad \bar{\nabla}_X Y = \nabla_X Y - \frac{1}{r}\bar{g}(X,Y)\partial_r.$$

For $X \in T\bar{M}$ and $Y \in TM \subset \pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)\times_{\mathrm{SO}(n+1)}\mathbb{R}^{n+1}$ we have

$$\psi^{-1}(\bar{\nabla}_{\partial_r}\psi(\partial_r)) = \psi^{-1}(\bar{\nabla}_{\partial_r}\partial_r) = 0 \overset{!}{=} [h, d((0..0,1)^t)(\partial_r) + \bar{\omega}(dh\partial_r)(0..0,1)^t] = [h,\bar{\omega}(dh\partial_r)(0..0,1)^t]$$

and

$$\begin{aligned}\psi^{-1}(\bar{\nabla}_{\partial_r}\psi(Y)) &= \psi^{-1}(\bar{\nabla}_{\partial_r}\frac{1}{ar}ar\psi(Y)) = \psi^{-1}(\frac{1}{ar}\bar{\nabla}_{\partial_r}ar\psi(Y) + (\partial_r\frac{1}{ar})ar\psi(Y))\\ &= \psi^{-1}(\frac{1}{ar}\frac{1}{r}(ar\psi(Y)) - \frac{1}{ar^2}ar\psi(Y)) = 0\\ &\overset{!}{=} [h, 0 + \bar{\omega}(dh\partial_r)(Y_1,..,Y_n,0)^t]\end{aligned}$$

and thus $\bar{\omega}(dh\partial_r) = 0$. Furthermore $X = [\tilde{h}, ar(X_1,..,X_n,0)^t] = [\tilde{h}, ar\eta(dhX)]$ and we get

$$\psi^{-1}(\bar{\nabla}_X\psi(\partial_r)) = \psi^{-1}(\bar{\nabla}_X\partial_r) = \psi^{-1}(\frac{1}{r}X) = \psi^{-1}([\tilde{h}, a\eta(dhX)]) = [h, a\eta(dhX)],$$

proving $a\eta = \bar{\omega}\cdot\partial_r$. Since $\psi(Y) = [\tilde{h},(Y_1,..,Y_n,0)^t]$, we have $\bar{g}(X, ar\psi(Y)) = a^2r^2\eta(dhX)^t\cdot(Y_1,..,Y_n)^t$. Furthermore we have

$$\nabla_X ar\psi(Y) = [\tilde{h}, ar(d(Y_1,..Y_n,0)^t(X) + ar(\omega(dhX)(Y_1,..,Y_n)^t,0)^t]$$

and obtain

$$\begin{aligned}&\psi^{-1}(\bar{\nabla}_X\psi(Y)) = \psi^{-1}(\frac{1}{ar}\bar{\nabla}_X ar\psi(Y)) = \psi^{-1}(\frac{1}{ar}\nabla_X ar\psi(Y) - \frac{1}{ar}\frac{1}{r}\bar{g}(X, ar\psi(Y))\partial_r)\\ &= \psi^{-1}([\tilde{h}, d(Y_1,..,Y_n,0)^t(X) + (\omega(dhX)(Y_1,..,Y_n)^t,0)^t - a\eta(dhX)^t(Y_1,..,Y_n,0)^t(0,..,0,1)^t]).\end{aligned}$$

Combining all these results yields

$$\bar{\omega} = \begin{bmatrix} \omega & a\eta \\ -a\eta^t & 0 \end{bmatrix}.$$

If one changes the orientation of $\bar{M}$ (a local SO($\bar{M}$) frame is then given by $(\frac{1}{ar}X_1, ..., \frac{1}{ar}X_2, -\partial_r)$), we obtain the alternative connection form

$$\begin{bmatrix} \omega & -a\eta \\ a\eta^t & 0 \end{bmatrix}.$$

For a ∇-Killing spinor on M with real Killing number α, we thus choose the cone constant $a = -2\alpha$ for $\alpha < 0$ and $a = 2\alpha$ for $\alpha > 0$. Hence, the cone *depends* on the Killing number and the construction only makes sense if $\alpha \in \mathbb{R}\backslash\{0\}$, as we had assumed from the beginning. In particular, the results cannot be applied to ∇-parallel spinors ($\alpha = 0$). The pullback of the connection $\tilde{\omega}$ under the projection $\pi : \bar{M} \to M$ is the same as the connection $\bar{\omega}$ on $\bar{M}$, thus their holonomy groups $Hol(\tilde{\omega})$ and $Hol(\bar{\omega})$ are the same. Since the second Stiefel-Whitney class of $\bar{M} = M \times \mathbb{R}$ is given by [Th52, p.142]

$$w_2(\bar{M}) \;=\; w_2(M) + w_2(\mathbb{R}) + w_1(M) \otimes w_1(\mathbb{R}),$$

we conclude that $\bar{M}$ is spin, since we assumed M to be spin.
Let us now have a closer look at spinors on M and $\bar{M}$. A parallel spinor of $(\bar{M}, \bar{\omega})$ is the same as a trivial factor of the action of the holonomy group $Hol(\bar{\omega}) = Hol(\tilde{\omega})$ on Δ_{n+1}. A Killing spinor on (M,ω) corresponds to a trivial factor of the action of the same group on the space Δ_n.
For $n = \dim(M)$ odd, the spin representation splits into $\Delta_{n+1} = \Delta_n^+ \oplus \Delta_n^-$. Changing the orientation of $\bar{M}$ (changing from negative to positive α and vice versa) means interchanging Δ_n^+ and Δ_n^-. Thus, a parallel spinor on $\bar{M}$ is either in Δ_n^+ or in Δ_n^-, giving either a Killing spinor with positive or with negative Killing number α.
For n even, we have $\Delta_n = \Delta_{n+1}$ and, by interchanging the orientation, we obtain for any parallel spinor in $\bar{M}$ one Killing spinor with positive, and one with negative Killing number α. We summarize these results in the following lemma:

Lemma 2.1. *For a Riemannian spin manifold (M, g) with metric connection ∇ with skew symmetric torsion T, consider the manifold $(\bar{M}, \bar{g})$ with connection $\bar{\nabla}$ with skew symmetric torsion $\bar{T}$ as constructed above. The following correspondence holds:*

- *If $n = \dim(M)$ is odd, any $\bar{\nabla}$-parallel spinor on $\bar{M}$ corresponds to a ∇-Killing spinor on M, with either positive or negative Killing number $\frac{1}{2}a$ or $-\frac{1}{2}a$.*
- *If n is even, any $\bar{\nabla}$-parallel spinor on $\bar{M}$ corresponds to a pair of ∇-Killing spinors on M with Killing number $\pm\frac{1}{2}a$.*

Remark 2.2. For $\dim M$ even, one can write down the bijection between Killing spinors with torsion with Killing numbers $\pm\alpha$ explicitly: If ϕ has Killing number α and decomposes into $\phi = \phi_+ + \phi_-$ in the spin bundle $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, then $\phi_+ - \phi_-$ is a Killing spinor with Killing number $-\alpha$. This is the same argument as in the Riemannian case [BFGK91, p.121].

Remark 2.3. The careful reader will have noticed that our cone is slightly more general than in [Bä93], where the computations are done for cone constant $a = 1$. This stems from the fact that in the Riemannian case, the Killing number is determined through $n = \dim M$ and Scal^g (remember that the manifold has to be Einstein), hence the cone can be normalized in such a way that $a = 1$. For our applications, this is too restrictive.

2.2 The cone correspondence for spinors

Let the cone $(\bar{M}, \bar{g})$ over M with Levi-Civita connection $\bar{\nabla}^{\bar{g}}$ carry a $\bar{G}$ structure and assume that there is a connection ∇ on M such that its lift $\bar{\nabla}$ to $\bar{M}$ with torsion $\bar{T}$ is the characteristic connection on $\bar{M}$ with respect to the given $\bar{G}$ structure.

Given a G structure on M, we shall construct an induced $\bar{G}$ structure on $\bar{M}$ in the following sections. We will see that the characteristic connection ∇^c on M (with torsion T^c) does *not* lift to the characteristic connection $\bar{\nabla}$ on $\bar{M}$ (with torsion $\bar{T}$, introduced by a connection ∇ on M with torsion T). In particular the lift $\overline{T^c}$ of the characteristic torsion to $\bar{M}$ is not the characteristic torsion on $\bar{M}$. So the tensor $T^c - T$ is not zero and will play an important role in the following. As introduced in the beginning of this section, we will use Killing spinors (not generalized Killing spinors) with torsion ϕ satisfying the equation

$$\nabla^s_X \phi \;=\; \alpha X \phi$$

for some Killing number $\alpha \in \mathbb{R} - \{0\}$ and some value of s, where $\nabla^s_X Y \;=\; \nabla^g_X Y + 2sT^c(X,Y)$. This definition includes the choice that we do not view a parallel spinor ($\alpha = 0$) as a special case of a Killing spinor. A priori, solutions of this equation with $\alpha \in \mathbb{C} - \mathbb{R}$ are conceivable, but we are not aware of any. In any event, the cone construction would not work for such an α.
The connection $\bar{\nabla}^s$ on $\bar{M}$ is then given by $\bar{\nabla}^s = \bar{\nabla}^{\bar{g}} + 2s\bar{T}$. We obtain the following correspondence between connections on $\bar{M}$ and connections on M:

Connections on M	Connections on $\bar{M}$
$\nabla^s = \nabla^g + 2sT^c$	$\bar{\nabla}^{\bar{g}} + 2s\overline{T^c} = \bar{\nabla}^s - 2s(\bar{T} - \overline{T^c})$
$\nabla^g + 2sT = \nabla^s + 2s(T - T^c)$	$\bar{\nabla}^s = \bar{\nabla}^{\bar{g}} + 2s\bar{T}$

A direct application of Lemma 2.1 implies:

Lemma 2.4. *For $\alpha \in \mathbb{R} - \{0\}$, we have the following correspondence between*

spinors on M	*spinors on $\bar{M}$*
$\nabla^s_X \phi = \alpha X \phi$	$\bar{\nabla}^s_X \phi - sX \lrcorner (\bar{T} - \overline{T^c})\phi = 0$
$\nabla^s_X \phi + sX \lrcorner (T - T^c)\phi = \alpha X \phi$	$\bar{\nabla}^s_X \phi = 0$

For $\dim(M)$ odd, there is one spinor on M with either $\alpha = \frac{1}{2}a$ or $\alpha = -\frac{1}{2}a$.
If $\dim(M)$ is even, there is a pair of spinors with Killing numbers $\alpha = \pm\frac{1}{2}a$ on M.
In particular for $s = \frac{1}{4}$ we obtain the following correspondence:

spinors on M	*spinors on $\bar{M}$*
$\nabla^c_X \phi = \alpha X \phi$	$\bar{\nabla}_X \phi = \frac{1}{4} X \lrcorner (\bar{T} - \overline{T^c})\phi$
$\nabla^c_X \phi = \alpha X \phi - \frac{1}{4} X \lrcorner (T - T^c)\ \phi$	$\bar{\nabla}_X \phi = 0$

In the following sections we look at the corresponding structures on $\bar{M}$, their classifications and the correspondences of spinors on M and $\bar{M}$.

3 Almost hermitian cones over almost contact manifolds

Let (M, g, ψ, η) be an n-dimensional metric almost contact structure. As in Section 2.1 we construct the twisted cone $\bar{M}$ over M and define an almost hermitian structure J on $\bar{M}$ via

$$J(ar\partial_r) := \xi, \qquad J(\xi) := -ar\partial_r \qquad \text{and} \qquad J(X) = -\psi(X) \text{ for } X \perp \xi, \partial_r.$$

The identity $\psi^2 = -\mathrm{Id} + \eta \otimes \xi$ immediately implies $J^2 = -\mathrm{Id}$.

Definition 3.1. If M admits a characteristic connection ∇^c with skew symmetric torsion T^c satisfying $\nabla^c\psi = \nabla^c\eta = 0$, we define a connection ∇ with skew symmetric torsion T

$$T \;:=\; T^c - 2a\eta \wedge F \text{ and thus } \nabla_X Y \;=\; \nabla^c_X Y - a(\eta \wedge F)(X, Y, .).$$

In particular: If the almost metric contact structure is Sasakian and the Killing number happens to satisfy $|\alpha| = 1/2$ (like in the Riemannian case), the cone is constructed with $a = 1$, and thus $T^c = \eta \wedge d\eta = 2a\eta \wedge F$ and $\nabla = \nabla^g$, the Levi-Civita connection. Thus, ∇ and T measure in some sense the difference to the Riemannian Sasakian case.
Although the role of T is clearly exposed in Section 2.2, this is not sufficient to determine T completely. Rather, the formula for T has to be found by trying a suitable Ansatz, the motivation for which comes precisely from the Riemannian case just described. Since T is unique, the definition is justified a posteriori by yielding the desired correspondence.

Theorem 3.2. *If (M, g, ψ, η) is an almost contact metric structure, $(\bar{M}, \bar{g}, J)$ is an almost hermitian manifold.*
If furthermore M admits a characteristic connection, consider the connection ∇ defined above. Then the appendant connection $\bar{\nabla}$ on $\bar{M}$ is almost complex, $\bar{\nabla}J = 0$.

Remark 3.3. This shows in particular that $\bar{\nabla}$ is the unique characteristic connection of $\bar{M}$ with respect to J. Furthermore, the theorem includes the claim that the existence of a characteristic connection for the almost contact metric structure on (M, g, ψ, η) suffices to imply that the induced almost hermitian structure on $\bar{M}$ does also admit a characteristic connection.

We first prove

Lemma 3.4. *On M, Definition* 3.1 *implies*

$$(\nabla_Y\psi)X = ag(Y, X)\xi - a\eta(X)Y, \tag{II.3}$$

and we have

a) $a\psi(X) = -\nabla_X\xi$,

b) ξ *is a Killing vector field,* $g(\nabla_Y\xi, X) = -g(\nabla_X\xi, Y)$ *and thus its integral curves are geodesics,*

c) $d\eta = 2aF + \xi \lrcorner T$.

Proof of Lemma 3.4. Using the definition $\nabla = \nabla^c - a\eta \wedge F$ with the equation $\nabla^c\psi = 0$, we directly compute $(\nabla_Y\psi)X = ag(Y, X)\xi - a\eta(X)Y$. Identity (II.3) and $\psi(\xi) = 0$ imply for $X \in TM$

$$aX - ag(X, \xi)\xi \;=\; -(\nabla_X\psi)\xi \;=\; \nabla_X(\psi(\xi)) - (\nabla_X\psi)\xi = \psi(\nabla_X\xi).$$

Since $\nabla_X\xi \perp \xi$, applying ψ yields

$$a\psi(X) \;=\; \psi(aX - ag(X, \xi)\xi) \;=\; -\nabla_X\xi.$$

Since $g(X,\psi(Y)) = -g(\psi(X),Y)$, we can conclude from equation (II.3) the statement b) of the lemma, which is also a consequence of Theorem 8.2 in [FI02]. For $X,Y \in TM$, we obtain with statement a)

$$\begin{aligned} d\eta(X,Y) &= X\eta(Y) - Y\eta(X) - \eta([X,Y]) \;=\; Xg(Y,\xi) - Yg(X,\xi) - g([X,Y],\xi) \\ &= g(\nabla_X Y,\xi) + g(Y,\nabla_X\xi) - g(\nabla_Y X,\xi) - g(X,\nabla_Y\xi) - g([X,Y],\xi) \\ &= T(X,Y,\xi) - g(Y,a\psi(X)) + g(X,a\psi(Y)) \;=\; T(X,Y,\xi) + 2aF(X,Y) \end{aligned}$$

which finishes the proof. □

Proof of Theorem 3.2. One easily checks that $\bar g(JX,JY) = \bar g(X,Y)$ for $X,Y \in T\bar M$ and thus J is an almost hermitian structure.
We have to show $\bar\nabla J = 0$, meaning $0 = \bar\nabla_Y(J(X)) - J(\bar\nabla_Y X)$. To do so, we distinguish the following cases:
If $X \in TM$, $X \perp \xi$ and $Y \in TM$ we have

$$\begin{aligned} \bar\nabla_Y(J(X)) - J(\bar\nabla_Y X) &= -\bar\nabla(\psi(X)) - J(\nabla_Y X - \frac{1}{r}\bar g(Y,X)\partial_r) \\ &= -\nabla_Y(\psi(X)) + \frac{1}{r}\bar g(Y,\psi(X))\partial_r - J(\nabla_Y X) + \frac{1}{ar^2}\bar g(Y,X)\xi \\ &= -(\nabla_Y\psi)(X) - \psi(\nabla_Y X) + a^2 r g(Y,\psi(X))\partial_r - J(\nabla_Y X) + ag(Y,X)\xi. \end{aligned}$$

With identity (II.3) and since $\eta(X) = 0$, $\psi(\xi) = 0$ we get

$$\begin{aligned} \bar\nabla_Y(J(X)) - J(\bar\nabla_Y X) &= -a\eta(X)Y - \psi(\nabla_Y X) + a^2 r g(Y,\psi(X))\partial_r - J(\nabla_Y X) \\ &= -\psi(\nabla_Y X + ag(Y,\psi(X))\xi) - J(ag(Y,\psi(X))\xi + \nabla_Y X), \end{aligned}$$

which is equal to zero if $\nabla_Y X + ag(Y,\psi(X))\xi$ is perpendicular to ξ and ∂_r. Obviously it is perpendicular to ∂_r. We have $g(\nabla_Y X + ag(Y,\psi(X))\xi,\xi) = 0$ if

$$0 \;=\; g(\nabla_Y X,\xi) + g(Y,a\psi(X)) \;=\; -g(X,\nabla_Y\xi) + g(Y,a\psi(X)) \;=\; g(X,a\psi(Y)) + g(Y,a\psi(X)) \;=\; 0.$$

If $X \in TM$, $X \perp \xi$ and $Y = \partial_r$ we have $\bar\nabla_Y(J(X)) - J(\bar\nabla_Y X) = \frac{1}{r}J(X) - J(\frac{1}{r}X) = 0$.
If $X = \xi$, $Y = \partial_r$ we get

$$\bar\nabla_Y(J(X)) - J(\bar\nabla_Y X) \;=\; \bar\nabla_{\partial_r}(-ar\partial_r) - J(\frac{1}{r}\xi) \;=\; -a\partial_r - ar\bar\nabla_{\partial_r}\partial_r + a\partial_r = 0.$$

Given $X = \xi$ and $Y = \xi$ we have

$$\bar\nabla_Y(J(X)) - J(\bar\nabla_Y X) \;=\; -\bar\nabla_\xi(ar\partial_r) - J(\nabla_\xi\xi - \frac{1}{r}\bar g(\xi,\xi)\partial_r) \;=\; -a\xi + a\xi = 0.$$

If $X = \xi$, $Y \in TM$, $Y \perp \xi$ we have

$$\bar\nabla_Y(J(X)) - J(\bar\nabla_Y X) \;=\; -\bar\nabla_Y(ar\partial_r) - J(\nabla_Y\xi - \frac{1}{r}\bar g(Y,\xi)\partial_r) = -aY + J(a\psi(Y)) \;=\; -aY + aY \;=\; 0.$$

Given $X = \partial_r$, $Y \perp \xi$, $Y \in TM$ we get

$$\bar\nabla_Y(J(X)) - J(\bar\nabla_Y X) \;=\; \bar\nabla_Y(\frac{1}{ar}\xi) - J(\frac{1}{r}Y) \;=\; -\frac{1}{ar}a\psi(Y) - J(\frac{1}{r}Y) = 0.$$

In the case $X = \partial_r$ and $Y = \xi$ we have

$$\bar\nabla_Y(J(X)) - J(\bar\nabla_Y X) \;=\; \bar\partial_\xi(\frac{1}{ar}\xi) - J(\frac{1}{r}\xi) \;=\; \frac{1}{ar}\nabla_\xi\xi - \frac{1}{ar^2}\bar g(\xi,\xi)\partial_r + a\partial_r \;=\; -a\partial_r + a\partial_r = 0.$$

The last case is given by $X = Y = \partial_r$. Then we have $\bar\nabla_{\partial_r}(\frac{1}{ar}\xi) = -\frac{1}{ar^2}\xi + \frac{1}{ar}\bar\nabla_{\partial_r}\xi = 0$. □

Let (M, g) be a Riemannian manifold such that the above constructed manifold $(\bar{M}, \bar{g})$ carries an almost hermitian structure J. We have $J(\partial_r) \perp \partial_r$. We consider the manifold $M = M \times \{1\} \subset \bar{M}$ and define for $X \in TM$: $\xi := aJ(\partial_r)$, $\eta(X) := g(X, \xi)$ and $\psi(X) := -J(X) + \bar{g}(J(X), \partial_r)\partial_r$. We get an almost contact structure on M:

$$\begin{aligned}\psi^2(X) &= -J(-J(X) + \bar{g}(J(X), \partial_r)\partial_r) + \bar{g}(J(-J(X) + \bar{g}(J(X), \partial_r)\partial_r), \partial_r)\partial_r \\ &= -X + \bar{g}(X, J(\partial_r))J(\partial_r) = -X + g(X, \xi)\xi = -X + \eta(X)\xi\end{aligned}$$

and

$$\begin{aligned}g(\psi(X), \psi(Y)) &= \frac{1}{a^2}\bar{g}(-J(X) + \bar{g}(J(X), \partial_r)\partial_r, -J(Y) + \bar{g}(J(Y), \partial_r)\partial_r) \\ &= \frac{1}{a^2}(\bar{g}(J(X), J(Y)) - \bar{g}(X, J\partial_r)\bar{g}(Y, J(\partial_r))) \;=\; g(X, Y) - \eta(X)\eta(Y).\end{aligned}$$

Conversely to Theorem 3.2 we prove:

Theorem 3.5. *Consider the manifold $\bar{M}$ equipped with a connection $\bar{\nabla}$ with skew symmetric torsion $\bar{T}$ being the lift of a connection ∇ with torsion T on M. If the connection $\bar{\nabla}$ is almost complex on $\bar{M}$, we have $(\nabla_X\psi)(Y) = ag(X, Y)\xi - a\eta(Y)X$ and thus the characteristic connection ∇^c on $M = M \times \{1\}$ has torsion $T^c = T + 2a\eta \wedge F$.*

Proof. Given a parallel J on $\bar{M}$ we have for $X, Y \in TM$

$$\begin{aligned}(\nabla_X\psi)(Y) =& \bar{\nabla}_X(\psi(Y)) + \bar{g}(X, \psi(Y))\partial_r - \psi(\bar{\nabla}_X Y + \bar{g}(X, Y)\partial_r) \\ =& -\bar{\nabla}_X J(Y) + \underbrace{(X\bar{g}(J(Y), \partial_r))\partial_r}_{\bar{g}(\bar{\nabla}_X J(Y), \partial_r)\partial_r + \bar{g}(J(Y), X)\partial_r} + \bar{g}(J(Y), \partial_r)\bar{\nabla}_X\partial_r - \bar{g}(X, J(Y))\partial_r \\ &+ J(\bar{\nabla}_X Y) + \bar{g}(X, Y)J(\partial_r) - \bar{g}(J(\bar{\nabla}_X Y), \partial_r)\partial_r \\ =& -\bar{g}(Y, J(\partial_r))X + \bar{g}(X, Y)J(\partial_r) = ag(X, Y)\xi - a\eta(Y)X.\end{aligned}$$

Writing $\nabla^c = \nabla + \frac{1}{2}(T^c - T)$ and using equation (II.3), the condition $\nabla^c\psi = \nabla^c\eta = 0$ leads to the following two conditions for $X, Y, Z \in TM$

$$\frac{1}{2}(T^c - T)(X, \psi(Y), Z) + \frac{1}{2}(T^c - T)(X, Y, \psi(Z)) = ag(X, Y)\eta(Z) - ag(X, Z)\eta(Y)$$

and

$$\frac{1}{2}(T^c - T)(X, Y, \xi) = (\nabla_X\eta)(Y) = g(X, \nabla_Y\xi) = F(Y, X).$$

Defining $T^c = T + 2a\eta\wedge F$, T^c satisfies these two conditions and thus ∇^c is the unique ([AFH13]) characteristic connection of the almost contact metric structure. $\square$

Remark 3.6. Conversely given an almost complex structure on M and $\partial_r = \xi$, $\eta = \bar{g}(., \xi)$ we do not have an almost contact structure on $\bar{M}$ since $d\eta = 0$.

From now on we assume that M and $\bar{M}$ admit an almost contact structure and an almost hermitian structure, respectively, both admitting characteristic connections ∇^c and $\bar{\nabla}$ as introduced above.

3.1 The classification of metric almost contact structures and the corresponding classification of almost hermitian structures on the cone

We look at the classification of the geometric structures on $\bar{M}$ and M. We first prove the following two lemmata.

Lemma 3.7. *The Nijenhuis tensor $\bar{N}$ of the almost hermitian structure on $\bar{M}$ restricted to TM and the Nijenhuis tensor N of the almost contact structure on M are related via $a^2r^2N = \bar{N}$. Furthermore, the following conditions are equivalent:*

- $\partial_r \lrcorner \bar{N} = 0$,
- $d\eta(X, \psi Y) + d\eta(\psi X, Y) = 0$ *on* TM,
- $\xi \lrcorner N = 0$.

In particular $N = 0$ if and only if $\bar{N} = 0$.

Remark 3.8. In [HTY12], T. Houri, H. Takeuchi, and Y. Yasui considered hermitian manifolds $\bar{M}$ with a vanishing Nijenhuis tensor $\bar{N}$. They showed that in this case $N = 0$ and thus M is a normal almost contact manifold, which also is an immediate consequence of Lemma 3.7.

Remark 3.9. Since $N = 0$ if and only if $\bar{N} = 0$, the condition $\bar{N} = 0$ is sometimes used for the definition of an almost contact metric manifold to be normal (see for example [CG90]).

Proof of Lemma 3.7. Since we have

$$\bar{g}((\bar{\nabla}^{\bar{g}}_X J)Y, Z) \;=\; \bar{g}((\bar{\nabla}_X J)Y + \frac{1}{2}(\bar{J}\bar{T}(X, Y) - \bar{T}(X, JY)), Z) \;=\; -\frac{1}{2}(\bar{T}(X, Y, JZ) + \bar{T}(X, JY, Z)),$$

the Nijenhuis tensor of $\bar{M}$ is given by

$$\begin{aligned}\bar{N}(X, Y, Z) \;&=\; \bar{g}((\bar{\nabla}^{\bar{g}}_X J)(JY), Z) - \bar{g}((\bar{\nabla}^{\bar{g}}_Y J)(JX), Z) + \bar{g}((\bar{\nabla}^{\bar{g}}_{JX} J)(Y), Z) - \bar{g}((\bar{\nabla}^{\bar{g}}_{JY} J)(X), Z)\\ &=\; \bar{T}(X, Y, Z) - \bar{T}(JX, JY, Z) - \bar{T}(JX, Y, JZ) - \bar{T}(X, JY, JZ),\end{aligned}$$

whereas the Nijenhuis tensor on M is

$$\begin{aligned}N(X, Y, Z) \;&= g((\nabla^g_X \psi)(\psi(Y)) - (\nabla^g_Y \psi)(\psi(X)) + (\nabla^g_{\psi(X)} \psi)(Y) - (\nabla^g_{\psi(Y)} \psi)(X), Z)\\ &\quad + \eta(X) g(\nabla^g_Y \xi, Z) - \eta(Y) g(\nabla^g_X \xi, Z).\end{aligned}$$

Identity (II.3) implies

$$g((\nabla^g_X \psi)(Y), Z) \;=\; a g(X, Y)\eta(Z) - a g(X, Z)\eta(Y) - \frac{1}{2}(T(X, \psi(Y), Z) + T(X, Y, \psi(Z)))$$

and hence we obtain for $N(X, Y, Z) =$

$$\begin{aligned}& a g(X, \psi(Y))\eta(Z) - \frac{1}{2}T(X, \psi^2(Y), Z)) - \frac{1}{2}T(X, \psi(Y), \psi(Z))\\ & - a g(Y, \psi(X))\eta(Z) + \frac{1}{2}T(Y, \psi^2(X), Z)) + \frac{1}{2}T(Y, \psi(X), \psi(Z))\\ & a g(\psi(X), Y)\eta(Z) - a g(\psi(X), Z)\eta(Y) - \frac{1}{2}T(\psi(X), \psi(Y), Z)) - \frac{1}{2}T(\psi(X), Y, \psi(Z))\\ & - a g(\psi(Y), X)\eta(Z) + a g(\psi(Y), Z)\eta(X) + \frac{1}{2}T(\psi(Y), \psi(X), Z)) + \frac{1}{2}T(\psi(Y), X, \psi(Z))\\ & + \eta(X) g(\nabla^c_Y \xi, Z) - \frac{1}{2}\eta(X) T^c(Y, \xi, Z) - \eta(Y) g(\nabla^c_X \xi, Z) + \frac{1}{2}\eta(Y) T^c(X, \xi, Z),\end{aligned}$$

which is the same as

$$\begin{aligned}&= T(X, Y, Z) - \frac{1}{2}\eta(Y) T(X, \xi, Z) - \frac{1}{2}\eta(X) T(\xi, Y, Z) - T(X, \psi(Y), \psi(Z))\\ &\quad - T(\psi(X), Y, \psi(Z)) - T(\psi(X), \psi(Y), Z) - a\eta(Y) g(\psi(X), Z) + a\eta(X) g(\psi(Y), Z)\\ &\quad - \frac{1}{2}\eta(X) T(Y, \xi, Z) - \eta(X) a F(Z, Y) + \frac{1}{2}\eta(Y) T(X, \xi, Z) + \eta(Y) a F(Z, X).\end{aligned}$$

For $X \in TM$ we have $J(X) + \eta(X)ar\partial_r = J(X - \eta(X)\xi) = -\psi(X - \eta(X)\xi) = -\psi(X)$. Since $\partial_r \lrcorner \bar{T} = 0$ for $X, Y, Z \in TM$ we get

$$\bar{T}(J(X), Y, Z) = -a^2r^2T(\psi(X), Y, Z) \tag{II.4}$$

and also $\bar{T}(J(X), J(Y), Z) = a^2r^2T(\psi(X), \psi(Y), Z)$ etc. With this result we have

$$N(X, Y, Z) = \frac{1}{a^2r^2}(\bar{T}(X, Y, Z) - \bar{T}(JX, JY, Z) - \bar{T}(JX, Y, JZ) - \bar{T}(X, JY, JZ))$$

and thus we get the desired result $\bar{N}(X, Z, Z) = a^2r^2N(X, Y, Z)$ for $X, Y, Z \in TM$.
By definition of the Nijenhuis tensor we have $\partial_r \lrcorner \bar{N} = 0$ if and only if for $X, Y \in TM$

$$0 = \bar{T}(\xi, JX, Y) + \bar{T}(\xi, X, JY) \Longleftrightarrow 0 = T(\xi, \psi X, Y) + T(\xi, X, \psi Y).$$

The relations $\xi \lrcorner T = d\eta - 2aF$ and

$$F(\psi X, Y) + F(X, \psi Y) \; = \; g(\psi X, \psi Y) + g(X, \psi^2 Y) \; = \; 0$$

imply that $\partial_r \lrcorner \bar{N} = 0$ holds if and only if $d\eta(\psi X, Y) + d\eta(X, \psi Y) = 0$. In [FI02] the identity $N(X, Y, \xi) = d\eta(X, Y) - d\eta(\psi X, \psi Y)$ is proved and we get

$$N(\psi X, Y, \xi) \; = \; d\eta(\psi X, Y) + d\eta(X, \psi Y) - \eta(X)d\eta(\xi, \psi Y).$$

The identity $\xi \lrcorner T = d\eta - 2aF$ implies $\xi \lrcorner d\eta = 0$ and thus $d\eta(\psi X, Y) + d\eta(X, \psi Y) = 0$ if and only if $N(\psi X, Y, \xi) = 0$. Since N is skew symmetric we have $N(\xi, Y, \xi) = 0$ and thus $N(\psi X, Y, \xi) = 0$ is equivalent to $\xi \lrcorner N = 0$. □

Lemma 3.10. *For $Z \in T\bar{M}$ let Z_M be the projection of Z onto TM. Then we have*

$$\delta\omega(Z) = -(\delta F - a(n-1)\eta)(Z_M).$$

Proof. For $X, Y, Z \in T\bar{M}$ we have

$$\begin{aligned}(\bar{\nabla}^{\bar{g}}_X \omega)(Y, Z) \; &= \; (\bar{\nabla}_X \omega)(Y, Z) - \omega(-\frac{1}{2}\bar{T}(X, Y), Z) - \omega(Y, -\frac{1}{2}\bar{T}(X, Z)) \\ &= \; \frac{1}{2}(\bar{T}(X, JY, Z) + \bar{T}(X, Y, JZ)).\end{aligned}$$

For a local ONB $\{e_1, .., e_n = \xi\}$ of TM we get the local ONB $\{\bar{e}_1 = \frac{1}{ar}e_1, .., \bar{e}_n = \frac{1}{ar}e_n, \bar{e}_{n+1} = \partial_r\}$ of $T\bar{M}$. In this basis and for $Z \in T\bar{M}$ we compute

$$\delta\omega(Z) = -\sum_{i=1}^{n+1}(\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\omega)(\bar{e}_i, Z) = -\frac{1}{2}\sum_{i=1}^{n-1}\bar{T}(\frac{1}{ar}e_i, \frac{1}{ar}Je_i, Z) - \frac{1}{2}\bar{T}(\frac{1}{ar}\xi, -\partial_r, Z) - \frac{1}{2}\bar{T}(\partial_r, J\partial_r, Z).$$

Since $\partial_r \lrcorner \bar{T} = 0$, with equation (II.4) and the fact that $\psi(e_n) = 0$, we have

$$\begin{aligned}\delta\omega(Z) &= \frac{1}{2}\sum_{i=1}^{n-1}T(e_i, \psi e_i, Z_M) = \frac{1}{2}\sum_{i=1}^{n-1}(T^c(e_i, \psi e_i, Z_M) - 2a(\eta \wedge F)(e_i, \psi e_i, Z_M)) \\ &= \frac{1}{2}\sum_{i=1}^{n-1}(T^c(e_i, \psi e_i, Z_M) - 2a\eta(Z_M)F(e_i, \psi e_i)) = \frac{1}{2}\sum_{i=1}^{n}T^c(e_i, \psi e_i, Z_M) + a\eta(Z_M)(n-1) \\ &= -(\delta F - a(n-1)\eta)(Z_M),\end{aligned}$$

finishing the proof. □

We consider the Gray-Hervella classification [GH80] of almost hermitian structures, given in Section 2 of Chapter I. Since we want to work with characteristic connections, we will only consider structures of class $\chi_1 \oplus \chi_3 \oplus \chi_4$. We first translate the conditions of this classification for the almost hermitian structure on $\bar{M}$ to conditions of the almost contact structure on M. For the discussion of the classification of almost contact structures and the correspondences to the classification of almost hermitian structures see Theorem 3.12.

Theorem 3.11. *We have the following correspondence between Gray-Hervella classes of almost hermitian structures on the cone $\bar{M}$ and defining relations of almost contact metric structures on M:*

Class of $\bar{M}$	*defining relation on $\bar{M}$*	*corresponding relation on M*
Kähler	$\bar{\nabla}^{\bar{g}} J = 0$	$(\nabla^g_X F)(Y,Z) = a\eta(Y)g(X,Z)$ $-a\eta(Z)g(X,Y)$
χ_3	$\delta\omega = \bar{N} = 0$	$N = 0$, $\delta F = a(n-1)\eta$
χ_4	$(\bar{\nabla}^{\bar{g}}_X \omega)(Y,Z) = \frac{-1}{n-1}[\bar{g}(X,Y)\delta\omega(Z)$ $-\bar{g}(X,Z)\delta\omega(Y) - \bar{g}(X,JY)\delta\omega(JZ)$ $+\bar{g}(X,JZ)\delta\omega(JY)]$	$(\nabla^g_X F)(Y,Z) = \frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y)$ $-g(X,Y)\eta(Z))$
$\chi_1 \oplus \chi_3$	$\delta\omega = 0$	$\delta F = a(n-1)\eta$
$\chi_3 \oplus \chi_4$	$\bar{N} = 0$	$N = 0$

Furthermore, a structure on $\bar{M}$ is never nearly Kähler (of class χ_1) nor of mixed class $\chi_1 \oplus \chi_4$.

Proof. We have

$$a\eta(Y)g(X,Z) - a\eta(Z)g(X,Y) = a\eta \wedge F(X,Y,\psi Z) + a\eta \wedge F(X,\psi Y,Z).$$

Kähler case: Since the characteristic connection on $\bar{M}$ is unique, we have the following equivalences

$$\bar{\nabla}^{\bar{g}} J = 0 \Leftrightarrow \bar{\nabla}^{\bar{g}} = \bar{\nabla} \Leftrightarrow \bar{T} = 0 \Leftrightarrow T = 0 \Leftrightarrow T^c = 2a\eta \wedge F.$$

For a metric connection $\tilde{\nabla}$ with skew symmetric torsion $\tilde{T}$ on M one calculates

$$(\tilde{\nabla}_X F)(Y,Z) \;=\; (\nabla^g_X F)(Y,Z) - \frac{1}{2}\tilde{T}(X,\psi Y,Z) - \frac{1}{2}\tilde{T}(X,Y,\psi Z).$$

Thus, $T^c = 2a\eta \wedge F$ implies $(\nabla^g_X F)(Y,Z) = a\eta \wedge F(X,Y,\psi Z) + a\eta \wedge F(X,\psi Y,Z)$ and conversely the condition $(\nabla^g_X F)(Y,Z) = a\eta \wedge F(X,Y,\psi Z) + a\eta \wedge F(X,\psi Y,Z)$ yields

$$(\tilde{\nabla}_X F)(Y,Z) = (a\eta \wedge F - \frac{1}{2}\tilde{T})(X,\psi Y,Z) + (a\eta \wedge F - \frac{1}{2}\tilde{T})(X,Y,\psi Z).$$

The uniqueness of the characteristic connection ∇^c on M thus implies $T^c = 2a\eta \wedge F$.

Case χ_3: Consider an almost hermitian structure on $\bar{M}$ of class χ_3 defined by $\delta\omega = \bar{N} = 0$. With Lemma 3.7 and 3.10 we have $\bar{N} = \delta\omega = 0$ if and only if $N = 0$ and $\delta F - a(n-1)\eta = 0$.

Case χ_4: The defining relation for the class χ_4 of an almost hermitian manifold $\bar{M}$

$$(\bar{\nabla}^{\bar{g}}_X \omega)(Y,Z) = \frac{-1}{n-1}[\bar{g}(X,Y)\delta\omega(Z) - \bar{g}(X,Z)\delta\omega(Y) - \bar{g}(X,JY)\delta\omega(JZ) + \bar{g}(X,JZ)\delta\omega(JY)]$$

translates with Lemma 3.10 for $X, Y, Z \in T\bar{M}$ into

$$\frac{1}{2}\bar{T}(X,Y,JZ) + \frac{1}{2}\bar{T}(X,JY,Z) =$$
$$\frac{1}{n-1}[\bar{g}(X,Y)(\delta F(Z_M) - a(n-1)\eta(Z_M)) - \bar{g}(X,Z)(\delta F(Y_M) - a(n-1)\eta(Y_M))$$
$$- \bar{g}(X,JY)(\delta F((JZ)_M) - a(n-1)\eta((JZ)_M)) + \bar{g}(X,JZ)(\delta F((JY)_M) - a(n-1)\eta((JY)_M))].$$

For $X \in T\bar{M}$ we have $\bar{g}(\partial_r, JX) = -\bar{g}(J\partial_r, X) = -\bar{g}(J\partial_r, X_M) = -a^2r^2 g(\frac{1}{ar}\xi, X_M) = -ar\eta(X_M)$ and for $X \in TM$ we have $(JX)_M = -\psi X$.
In the case where $X = \partial_r$ and $Y, Z \in TM$, the defining relation is equivalent to

$$0 = ar\eta(Y)(\delta F(-\psi Z) - a(n-1)\eta(-\psi Z)) - ar\eta(Z)(\delta F(-\psi Y) - a(n-1)\eta(-\psi Y)).$$

This is satisfied if and only if $0 = (\eta(Y)\delta F(\psi Z) - \eta(Z)\delta F(\psi Y)) = \eta \wedge (\delta F \circ \psi)(Y,Z)$. Taking $Y = \xi$ we receive the condition $F \circ \psi = 0$, which obviously is sufficient too.
If $X = Y = \partial_r$, $Z \in TM$ the defining relation leads to

$$0 = \delta F(Z) - a(n-1)\eta(Z) - ar\eta(Z)(\delta F(\frac{1}{ar}\xi) - \frac{n-1}{r}),$$

which is the same as $0 = \delta F(Z) - \eta(Z)\delta F(\xi) = -\delta F(\psi^2 Z)$, already being satisfied if $\delta F \circ \psi = 0$.
The case $Y = Z = \partial_r$ leads to $0 = 0$.
Given $Y = \partial_r$ and $X, Z \in TM$ we get

$$\frac{1}{2ar}\bar{T}(X,\xi,Z) = \frac{1}{n-1}[ar\eta(X)\delta F(-\psi(Z)) - a^2r^2F(X,Z)(\delta F(\frac{1}{ar}\xi) - a(n-1)\frac{1}{ar})].$$

Since we already have the condition $\delta F \circ \psi = 0$ this is equivalent to

$$d\eta(X,Z) - 2aF(X,Z) = (\xi \lrcorner T)(X,Z) = \frac{2}{n-1}F(X,Z)(\delta F(\xi) - a(n-1)).$$

This is the same as $d\eta = \frac{2}{n-1}\delta F(\xi)F$. At last we look at $X, Y, Z \in TM$. Again we already have $\delta F \circ \psi = 0$

$$\begin{aligned}
&-\frac{1}{2}T(X,Y,\psi Z) - \frac{1}{2}T(X,\psi Y,Z) \\
&= \frac{1}{n-1}[g(X,Y)(\delta F(Z) - a(n-1)\eta(Z)) - g(X,Z)(\delta F(Y) - a(n-1)\eta(Y))] \\
&= g(X,Y)(\frac{\delta F}{n-1} - a\eta)(Z) - g(X,Z)(\frac{\delta F}{n-1} - a\eta)(Y).
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
-\frac{1}{2}(T(X,Y,\psi Z) + T(X,\psi Y,Z)) &= -\frac{1}{2}T^c(X,Y,\psi Z) - \frac{1}{2}T^c(X,\psi Y,Z) \\
&\quad + a\eta(X)F(Y,\psi Z) + a\eta(Y)F(\psi Z,X) \\
&\quad + a\eta(X)F(\psi Y,Z) + a\eta(Z)F(X,\psi Y) \\
&= -(\nabla^g_X F)(Y,Z) + a\eta(Y)g(\psi Z,\psi X) + a\eta(Z)g(X,\psi^2 Y) \\
&= -(\nabla^g_X F)(Y,Z) + a\eta(Y)g(Z,X) - a\eta(Z)g(X,Y).
\end{aligned}$$

Thus we get the equation

$$(\nabla^g_X F)(Y,Z) = g(X,Z)\frac{\delta F}{n-1}(Y) - g(X,Y)\frac{\delta F}{n-1}(Z).$$

Since $\delta F \circ \psi = 0$ we have $\delta F = \delta F(\xi)\eta$ and obtain

$$(\nabla^g_X F)(Y,Z)\eta(Y) - g(X,Y)(\frac{\delta F(\xi)}{n-1} + 2a)\eta(Z) = \frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y) - g(X,Y)\eta(Z)).$$

We summarize this result: An almost hermitian structure on $\bar{M}$, given by an almost contact structure on M is of class χ_4 if and only if

$$(\nabla^g_X F)(Y,Z) = \frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y) - g(X,Y)\eta(Z)), \quad \delta F \circ \psi = 0 \quad \text{and} \quad d\eta = 2\frac{\delta F(\xi)}{n-1}F.$$

The first condition implies the others: For some local orthonormal basis $e_1, \dots, e_n = \xi$ of TM we have

$$\begin{aligned}\delta F(X) &= -\sum_{i=1}^n (\nabla^g_{e_i} F)(e_i, X) = -\sum_{i=1}^n \frac{\delta F(\xi)}{n-1}(g(e_i,X)\eta(e_i) - \eta(X)) \\ &= -\frac{\delta F(\xi)}{n-1}(-n\eta(X) + \eta(X)) = \delta F(\xi)\eta(X)\end{aligned}$$

and thus the condition $(\nabla^g_X F)(Y,Z) = \frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y) - g(X,Y)\eta(Z))$ implies $\delta F \circ \psi = 0$. Since ξ is a Killing vector field and thus $(\nabla^g_X F)(\xi, \psi Y) = -F(\nabla^g_X \xi, \psi Y) = g(\nabla^g_X \xi, Y)$ is skew symmetric in X and Y we have

$$d\eta(X,Y) = (\nabla^g_X \eta)(Y) - (\nabla^g_Y \eta)(X) = (\nabla^g_X F)(\xi, \psi Y) - (\nabla^g_Y F)(\xi, \psi X) = 2(\nabla^g_X F)(\xi, \psi Y)$$

and with condition $(\nabla^g_X F)(Y,Z) = \frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y) - g(X,Y)\eta(Z))$ we already get $d\eta = 2\frac{\delta F(\xi)}{n-1}F$.

Case $\chi_1 \oplus \chi_3$: The condition for a structure of class $\chi_1 \oplus \chi_3$ can be obtained directly from Lemma 3.10.

Case $\chi_3 \oplus \chi_4$: An almost hermitian structure on $\bar{M}$ is of class $\chi_3 \oplus \chi_4$ if and only if $\bar{N} = 0$. Due to Lemma 3.7, this is equivalent to $N = 0$.

Case $\chi_1 \oplus \chi_4$: The condition for an almost hermitian structure to be of class $\chi_1 \oplus \chi_4$ is the same as for the class χ_4, setting $X = Y$:

$$\begin{aligned}\frac{1}{2}\bar{T}(X,JX,Y) =& \frac{1}{n-1}[\bar{g}(X,X)(\delta F(Y_M) - a(n-1)\eta(Y_M)) - \bar{g}(X,Y)(\delta F(X_M) - a(n-1)\eta(X_M)) \\ &+ \bar{g}(X,JY)(\delta F((JX)_M) - a(n-1)\eta((JX)_M))].\end{aligned}$$

The equation is still linear in Y but not in X. We set $X = V + b\partial_r$ for $b \in \mathbb{R}$ and $V \in TM$:

$$\begin{aligned}\frac{1}{2}\bar{T}(V,JV,Y) + \frac{b}{2ar}\bar{T}(V,\xi,Y) =& \frac{1}{n-1}[(b^2 + a^2r^2 g(V,V))(\delta F(Y_M) - a(n-1)\eta(Y_M)) \\ &- (b\bar{g}(\partial_r, Y) + a^2r^2 g(V,Y_M))(\delta F(V) - a(n-1)\eta(V)) \\ &+ (\bar{g}(V,JY) - bar\eta(Y_M))(-\delta F(\psi V) + \frac{b}{ar}\delta F(\xi) - \frac{b(n-1)}{r})].\end{aligned}$$

This is satisfied for any b if and only if

$$\begin{aligned}\frac{1}{2}\bar{T}(V,JV,Y) =& \frac{1}{n-1}[a^2r^2 g(V,V)(\delta F(Y_M) - a(n-1)\eta(Y_M)) \\ &- a^2r^2 g(V,Y_M)(\delta F(V) - a(n-1)\eta(V)) + \bar{g}(V,JY)(-\delta F(\psi V))]\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2ar}\bar{T}(V,\xi,Y) =& \frac{1}{n-1}[-\bar{g}(\partial_r,Y)(\delta F(V)-a(n-1)\eta(V)) \\ &+\bar{g}(V,JY)(\frac{\delta F(\xi)}{ar}-\frac{(n-1)}{r})+ar\eta(Y_M)\delta F(\psi V)]\end{aligned} \tag{II.5}$$

and

$$0=\delta F(Y_M)-\eta(Y_M)\delta F(\xi)=\delta F(Y_M-\eta(Y_M)\xi)=-\delta F(\psi^2(Y_M)),$$

where the last equation is satisfied if and only if $\delta F\circ\psi=0$.
For $Y\in TM$ with the condition $\delta F\circ\psi=0$ equation (II.5) leads to

$$\frac{1}{2}T(\xi,V,Y)=F(V,Y)(\frac{\delta F(\xi)}{n-1}-a).$$

Since $\xi\lrcorner T=d\eta-2aF$ we have $d\eta=2\frac{\delta F(\xi)}{n-1}F$ and thus $dF=0$.
With Theorem 8.4 in [FI02] this implies $N=0$ and the structure is already of class χ_4. Thus a structure is never of class χ_1 or of mixed class $\chi_1\oplus\chi_4$. □

We now compare the result of Theorem 3.11 with the 12 classes of almost contact structures given in Section 1 of Chapter I. We just consider manifolds admitting a characteristic connection (recall that Theorem 1.3 of Chapter I formulates the criterion for its existence).

Theorem 3.12. *If the almost hermitian structure on $\bar{M}$ is*

- *of class χ_3, then the almost contact structure on M is of class $\mathcal{C}_3\oplus..\oplus\mathcal{C}_8$ but not of class $\mathcal{C}_3\oplus\mathcal{C}_4\oplus\mathcal{C}_5\oplus\mathcal{C}_7\oplus\mathcal{C}_8$ or of class $\mathcal{C}_6$.*
- *of class $\chi_1\oplus\chi_3$, then the almost contact structure on M is not of class $\mathcal{C}_1\oplus..\oplus\mathcal{C}_5\oplus\mathcal{C}_7\oplus..\oplus\mathcal{C}_{12}$ nor of class $\mathcal{C}_6$.*

The almost hermitian structure on $\bar{M}$ is

- *Kähler if and only if the almost contact structure on M is α-Sasaki (of class $\mathcal{C}_6$) and $\delta F(\xi)=a(n-1)$.*
- *of class χ_4 if and only if the almost contact structure on M is an α-Sasaki structure.*
- *of class $\chi_3\oplus\chi_4$ if and only if the almost contact structure on M is of class $\mathcal{C}_3\oplus..\oplus\mathcal{C}_8$ and there exists a characteristic connection.*

Furthermore the structure on M is Sasaki if and only if the almost hermitian structure on $\bar{M}$ is of class χ_4 with $\delta\omega(\xi)=(a-1)(n-1)$.

Proof. If the structure on $\bar{M}$ is of class χ_3, we have $N=0$ and thus the structure on M is of class $\mathcal{C}_3\oplus..\oplus\mathcal{C}_8$. Furthermore, $\delta F(\xi)=a(n-1)$ holds, but on $\mathcal{C}_3\oplus\mathcal{C}_4\oplus\mathcal{C}_5\oplus\mathcal{C}_7\oplus\mathcal{C}_8$ we have $\delta F(\xi)=0$ and a structure on M of class $\mathcal{C}_6$ implies a structure on $\bar{M}$ of class χ_4.
A structure on $\bar{M}$ of class $\chi_1\oplus\chi_3$ implies on M the relation $\delta F(\xi)\neq 0$, but on $\mathcal{C}_1\oplus..\oplus\mathcal{C}_5\oplus\mathcal{C}_7\oplus..\oplus\mathcal{C}_{12}$ we have $\delta F(\xi)=0$ and again a structure on M of class $\mathcal{C}_6$ implies a structure on $\bar{M}$ of class χ_4.
With Theorem 3.11, a structure on $\bar{M}$ is Kählerian if and only if $(\nabla^g_X F)(Y,Z)=a\eta(Y)g(X,Z)-a\eta(Z)g(X,Y)$ holds on M, which is equivalent for the almost contact structure to be of class $\mathcal{C}_6$ with $\delta F(\xi)=a(n-1)$.
The condition of Theorem 3.11 for a structure of class χ_4 on M is equivalent to the definition of an almost contact structure on $\bar{M}$ to be of class $\mathcal{C}_6$.
In $\mathcal{C}_3\oplus..\oplus\mathcal{C}_8$ we have $N=0$, which together with the existence of a characteristic connection

is equivalent to the property that the structure on $\bar{M}$ is of class $\chi_3 \oplus \chi_4$.
A structure on M is Sasaki if and only if it is of class $\mathcal{C}_6$ and $\delta F(\xi) = n - 1$. Due to Theorem 3.11 this is equivalent to the condition for the structure on $\bar{M}$ to be of class χ_4 with $\delta\omega(\xi) = (a-1)(n-1)$. □

Remark 3.13. If we construct $\bar{M}$ with $a = 1$, we obtain a Kählerian structure, and $(\nabla^g_X F)(Y, Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y)$ defines a Sasakian structure on M. This is the classical case treated by Bär in [Bä93].

3.2 Corresponding spinors on metric almost contact structures and their cones

We shall now work out in detail the abstract spinor correspondence stated in Lemma 2.4 for the case that M carries a metric almost contact structure. The following result serves as a preparation.

Lemma 3.14. *Given a metric almost contact structure with characteristic connection on M, the lift of $\eta \wedge F$ to its cone $\bar{M}$ is given by*

$$\frac{1}{a^3 r^3}(\partial_r \lrcorner \omega) \wedge \omega.$$

Proof. Since $\partial_r \lrcorner [\frac{1}{a^3r^3}(\partial_r \lrcorner \omega) \wedge \omega] = 0$ we just need to show the equality on TM. For $X, Y \in TM$ we have

$$F(X, Y) = g(X, \psi Y) = -\frac{1}{a^2r^2}\bar{g}(X, JY + \eta(Y)ar\partial_r) = -\frac{1}{a^2r^2}\omega(X, Y)$$

and

$$\eta(X) = g(X, arJ\partial_r) = \frac{1}{ar}\omega(X, \partial_r)$$

which proves $F = -\frac{1}{a^2r^2}\omega$ and $\eta = -\frac{1}{ar}\partial_r \lrcorner \omega$ on TM. □

We recall the definition of the connections

$$\nabla^s_X Y = \nabla^g_X Y + 2sT^c(X, Y) \text{ and } \bar{\nabla}^s_X Y = \bar{\nabla}^{\bar{g}}_X Y + 2s\bar{T}(X, Y)$$

for $s \in \mathbb{R}$ from the beginning of this chapter. Theorem 3.2 yields $T^c = T + 2a\eta \wedge F$ and since $\bar{T} = a^2r^2T$ and $\overline{T^c} = a^2r^2T^c$, we get $\overline{T^c} - \bar{T}$ as the lift of $2a^3r^2\eta \wedge F$ to $\bar{M}$. With Lemma 3.14 we obtain $\overline{T^c} - \bar{T} = \frac{2}{r}(\partial_r \lrcorner \omega) \wedge \omega$.

Theorem 3.15. *Assume that the almost contact metric manifold (M, g, ψ, η) admits a characteristic connection and is spin. Then there is for $\alpha = \frac{1}{2}a$ or $\alpha = -\frac{1}{2}a$:*

1. *A one to one correspondence between Killing spinors with torsion*

$$\nabla^s_X \phi = \alpha X \phi$$

on M and parallel spinors of the connection $\bar{\nabla}^s + \frac{4s}{r}(\partial_r \lrcorner \omega) \wedge \omega$ on $\bar{M}$ with cone constant a

$$\bar{\nabla}^s_X \phi + \frac{2s}{r}(X \lrcorner (\partial_r \lrcorner \omega) \wedge \omega)\phi = 0,$$

2. *A one to one correspondence between $\bar{\nabla}^s$-parallel spinors on $\bar{M}$ with cone constant a and spinors on M satisfying*

$$\nabla^s_X \phi - 2asX \lrcorner (\eta \wedge F)\phi = \alpha X \phi.$$

In particular, for $s = \frac{1}{4}$ we get the correspondence

spinors on M	*spinors on* $\bar{M}$
$\nabla^c_X\phi = \alpha X\phi$	$\bar{\nabla}_X\phi = -\frac{1}{2r}X\lrcorner((\partial_r\lrcorner\omega)\wedge\omega)\phi$
$\nabla^c_X\phi = \alpha X\phi + \frac{a}{2}X\lrcorner(\eta\wedge F)\phi$	$\bar{\nabla}_X\phi = 0$

Remark 3.16. Assuming a Killing spinor with torsion on M, we get the correspondence to a spinor on $\bar{M}$ as in case (1) of the Theorem. In particular this spinor is parallel in the cone direction.

Remark 3.17. Since $\bar{\nabla} = \bar{\nabla}^{\bar{g}} + \frac{1}{2}\bar{T}$ is the characteristic connection of the almost hermitian structure on $\bar{M}$, we can write
$$\bar{T} \;=\; \bar{N} + d\omega^J,$$
where $d\omega^J = d\omega\circ J$. Thus one can rewrite all equations above. For example the correspondence (1) of Theorem 3.15 is given with spinors on $\bar{M}$ satisfying

$$\bar{\nabla}^{\bar{g}}_X\phi + sX\lrcorner[\bar{N} + d\omega^J + \frac{2}{r}(\partial_r\lrcorner\omega)\wedge\omega]\phi \;=\; 0.$$

Equivalently, one can use the description of T^c on M given by $T^c = \eta\wedge d\eta + dF^\psi + N - \eta\wedge(\xi\lrcorner N)$ ([FI02]) to rewrite the second correspondence. Note that this also implies that $\bar{T} = \bar{N} + d\omega^J$ is the lift of

$$a^2r^2T = a^2r^2(T^c - 2a\eta\wedge F) = a^2r^2(\eta\wedge(d\eta - 2aF) + dF^\psi + N - \eta\wedge(\xi\lrcorner N))$$

to $\bar{M}$, in particular we have $\partial_r\lrcorner(\bar{N} + d\omega^J) = 0$.

3.3 Examples

In this Section, we shall discuss several examples of metric almost contact structures and the special spinor fields that exist on them and on their cones. In particular, we shall describe several situations where the cone carries a parallel spinor field for the characteristic connection $\bar{\nabla}$ of its almost hermitian structure.

Example 3.18. For a metric almost contact manifold (M, g, ψ, η), the deformation

$$g_t := tg + (t^2 - t)\eta\otimes\eta, \quad \xi_t := \frac{1}{t}\xi, \quad \eta_t := t\eta, \quad t > 0$$

is often used for different purposes and constructions (compare Example 1.10). Since Tanno used it in [Ta68] it is the so called *Tanno deformation.* It has the property that if the original manifold is K-contact or Sasaki, then the deformed manifold $(M, g_t, \xi_t, \eta_t, \psi)$ has again this property.
In [Be12, Cor.2.18] it was proved that almost any Sasakian η-Einstein manifold satisfying

$$Ric^g = \lambda g - \nu\eta\otimes\eta \text{ for some } \lambda, \nu\in\mathbb{R}$$

carries Killing spinors with torsion. On an Einstein-Sasaki manifold (M, g, ψ, η) of dimension $n = 2k+1 \geqslant 5$ this spinors are constructed as follows. Consider the one dimensional subbundles of the spinor bundle Σ_t of (M, g_t) defined by

$$L_1(\Sigma_t) := \{\phi\in\Sigma_t \mid \psi(X)\phi = -iX\phi\ \forall X\perp\xi\}, \quad L_2(\Sigma_t) := \{\phi\in\Sigma_t \mid \psi(X)\phi = iX\phi\ \forall X\perp\xi\}.$$

Define $\epsilon = \pm 1$ to be the number satisfying $e_1\psi(e_1)...e_k\psi(e_k)\xi\phi = \epsilon i^{k+1}\phi$ for a local orthonormal frame $e_1, \psi(e_1), .., e_k, \psi(e_k), \xi$ on M. Theorem 2.22 from [Be12] then states that the spinors

$\phi_1 \in L_1(\Sigma_t)$ and $\phi_2 \in L_2(\Sigma_t)$ are Killing spinors with torsion for $s_t = \frac{k+1}{4(k-1)}(\frac{1}{t} - 1)$ with Killing numbers

$$\beta_{1,t} = \frac{\epsilon}{2}\frac{2kt-(k+1)}{t(k-1)} = \frac{\epsilon}{2}(1-4s_t) \qquad \text{and} \qquad \beta_{2,t} = (-1)^{k+1}\beta_{1,t} \tag{$*$}$$

respectively. For $t = 1$, there is no deformation, and indeed the parameter s_t is then zero and the two spinors are just classical Riemannian Killing spinors. Since $(M, g_t, \xi_t, \eta_t, \psi)$ with fundamental 2-form F_t is Sasakian, the characteristic torsion of ∇^c is given by $T^c = \eta_t \wedge d\eta_t = 2\eta_t \wedge F_t$. Thus, the Killing equation

$$\nabla^{g_t}_X \phi_i + s_t(X \lrcorner T^c)\phi_i \;=\; \beta_{i,t} X\phi_i, \quad i = 1, 2$$

can equivalently be reformulated as

$$\nabla^{g_t}_X \phi_i + \frac{1}{4}(X \lrcorner T^c)\phi_i - (1-4s_t)\frac{1}{4}(X \lrcorner T^c)\phi_i \;=\; \beta_{i,t} X\phi_i.$$

If $1 - 4s_t = 0$, both Killing numbers $\beta_{i,t}$ vanish by equation $(*)$ and the Killing equation is reduced to $\nabla^c\phi_i = 0$ – the spinor fields ϕ_i are ∇^c-parallel and, as observed before, the cone construction is not possible. The condition $1 - 4s_t > 0$ is equivalent to $t > \frac{k+1}{2k}$ and we observe that in this case, the last equation is exactly of the form treated in Theorem 3.15, case (2) for $s = 1/4$ and $a = 2|\beta_{i,t}| = 1 - 4s_t > 0$. Recall that we know from Theorem 3.12 that the cone $(\bar{M}, \bar{g}_t)$ of the Tanno deformation is a locally conformally Kähler manifold (class χ_4). Hence, we can conclude from Theorem 3.15, case (2):

Theorem 3.19. *Let (M, g, ψ, η) be an Einstein Sasaki manifold of dimension $2k+1 \geqslant 5$. Consider its Tanno deformation $(M, g_t, \xi_t, \eta_t, \psi)$ for $t > \frac{k+1}{2k}$ and the cone $(\bar{M}, \bar{g}_t, J_t)$, constructed with cone constant $a = 1 - 4s_t > 0$, and endowed with the conformally Kähler structure described before. Then the two Killing spinors with torsion on $(M, g_t, \xi_t, \eta_t, \psi)$ induce each a spinor on the cone $(\bar{M}, \bar{g}_t, J_t)$ that is parallel with respect to its characteristic connection $\bar{\nabla}$.*

Although Killing spinors with torsion do exist on $(M, g_t, \xi_t, \eta_t, \psi)$ for $0 < t < \frac{k+1}{2k}$, Theorem 3.15, case (2) cannot be applied because the signs do not match. Of course, case (1) does still hold and therefore we obtain a spinor field satisfying a more complicated equation on $\bar{M}$. For $t = 1$ (meaning $s_t = 0$), Theorem 3.19 is the classical cone correspondence between Riemannian Killing spinors on Einstein-Sasaki manifolds and Riemannian parallel spinors on their cone [Bä93].

Example 3.20. The Heisenberg group H is defined to be the following Lie subgroup of $\mathrm{Gl}(5, \mathbb{R})$:

$$H := \left\{ \begin{pmatrix} 1 & u & v & z \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid u, v, x, y, z \in \mathbb{R} \right\}.$$

The vector fields $u_1 = \partial_u$, $u_2 = \partial_x + u\partial_z$, $u_3 = \partial_v$, $u_4 = \partial_y + v\partial_z$ and $u_5 = \partial_z$ form a basis of the left invariant vector fields. For $\rho > 0$ we consider the metric ([KV85])

$$g = \frac{1}{\rho}(du^2 + dx^2 + dv^2 + dy^2) + (dz - udx - vdy)^2$$

and get an orthonormal frame $e_1 = \sqrt{\rho}u_1$, $e_2 = \sqrt{\rho}u_2$, $e_3 = \sqrt{\rho}u_3$, $e_4 = \sqrt{\rho}u_4$ and $e_5 = u_5$. We consider the almost contact structures given by

$$F_1 := e_1 \wedge e_2 - e_3 \wedge e_4 \text{ and } F_2 := e_1 \wedge e_2 + e_3 \wedge e_4,$$

both with the same $\xi := e_5$. Becker-Bender calculates in [Be12] that the characteristic connection for both structures is given by its torsion $T^c = -\rho e_1 \wedge e_2 \wedge e_5 - \rho e_3 \wedge e_4 \wedge e_5$. She also proves that ϕ_1 and ϕ_2, defined via the equations

$$\psi_2(X)\phi_1 = -iX\phi_1 \ \forall X \perp \xi \text{ and } \psi_2(X)\phi_2 = iX\phi_2 \ \forall X \perp \xi,$$

where ψ_j is the $(1,1)$ tensor to the 2-form F_j for $j = 1, 2$, are Killing spinors with torsion for $s = -\frac{3}{4}$ with Killing number ρ and $-\rho$ respectively:

$$\nabla_X^{-\frac{3}{4}}\phi_1 = \rho X\phi_1 \text{ and } \nabla_X^{-\frac{3}{4}}\phi_2 = -\rho X\phi_2$$

for $\nabla^{-\frac{3}{4}} := \nabla^g - \frac{3}{2}T^c$.
On $\bar{M}$ we get two almost hermitian structures, constructed from F_1 and F_2: We consider the orthonormal basis on $\bar{M}$ given by $X_i := \frac{1}{ar}e_i$ for $i = 1..5$ and $X_6 := \partial_r$. Then the almost hermitian structures on $\bar{M}$ are given by

$$\Omega_1 = -X_1 \wedge X_2 + X_3 \wedge X_4 + X_5 \wedge X_6 \text{ and } \Omega_2 = -X_1 \wedge X_2 - X_3 \wedge X_4 + X_5 \wedge X_6.$$

We look at the corresponding characteristic connections $\bar{\nabla}^1$ and $\bar{\nabla}^2$, coming from the connections ∇^1 and ∇^2 with torsions $T^1 = T^c - 2a\eta \wedge F_1$ and $T^2 = T^c - 2a\eta \wedge F_2$ on M, and the s-dependent connections $\bar{\nabla}^{s,1} := \bar{\nabla}^g + 2s\bar{T}^1$ and $\bar{\nabla}^{s,2} := \bar{\nabla}^g + 2s\bar{T}^2$. The equivalence of the characteristic connections for F_1 and F_2 on M implies, that the connections $\bar{\nabla}^{s,i} + \frac{4s}{r}(\partial_r \lrcorner \Omega) \wedge \Omega$ are the same for $i = 1, 2$. With Theorem 3.15 we get for both orientations on $\bar{M}$, constructed with $a = 2\rho$, the existence of a spinor ϕ satisfying

$$\bar{\nabla}_X^{-\frac{3}{4},i}\phi - \frac{3}{2r}X \lrcorner ((\partial_r \lrcorner \Omega_i) \wedge \Omega_i)\phi = 0 \text{ for } i = 1, 2.$$

Thus we have two linear independent spinors on $\bar{M}$ satisfying this equation, which, due to Remark 3.17 is equivalent to

$$\bar{\nabla}_X^g\phi - \frac{3}{4}X \lrcorner [\bar{N}_i + d\Omega_i + \frac{2}{r}(\partial_r \lrcorner \Omega_i) \wedge \Omega_i]\phi = 0$$

where $\bar{N}_i$ denotes the Nijenhuis tensor of the almost hermitian structure Ω_i. We calculate the types of the structures F_1 and F_2 and with Theorem 3.12 we get immediately

Lemma 3.21. *The structure F_1 is of type $\mathcal{C}_7$ and the structure F_2 is of type $\mathcal{C}_6$.
Thus the almost hermitian structure on $\bar{M}$ induced by F_1 is of mixed type $\chi_3 \oplus \chi_4$ and the almost hermitian structure on $\bar{M}$ induced by F_2 is of type χ_4.*

Proof. With the given F_i we get in the basis defined above $\psi_i e_1 = -e_2$ for $i = 1, 2$ and $\psi_1 e_3 = e_4$, $\psi_2 e_3 = -e_4$; the other values of $\psi_k e_j$ we get from the skew symmetry of ψ_k. With the equation $(\nabla_X^g F_2)(Y, Z) = \frac{1}{2}T^c(X, \psi_2 Y, Z) + \frac{1}{2}T^c(X, Y, \psi_2 Z)$ and the fact that $T^c = -\rho e_1 \wedge e_2 \wedge e_5 - \rho e_3 \wedge e_4 \wedge e_5 = -\rho F_2 \wedge \eta$ we get

$$\begin{aligned}\delta F_k(\xi) &= -\sum_i (\nabla_{e_i}^g F_k)(e_i, \xi) = -\sum_i \frac{1}{2}T^c(e_i, \psi_k e_i, \xi) \\ &= \frac{1}{2}\rho\sum_i (e_1 \wedge e_2 \wedge e_5 + e_3 \wedge e_4 \wedge e_5)(e_i, \psi_k e_i, \xi) = \frac{1}{2}\rho\sum_i (e_1 \wedge e_2 + e_3 \wedge e_4)(e_i, \psi_k e_i)\end{aligned}$$

and thus we have $\delta F_1(\xi) = 0$ and $\delta F_2(\xi) = -2\rho$. We calculate

$$\begin{aligned}(\nabla_X^g F_k)(Y, Z) &= \frac{1}{2}T^c(X, \psi_k Y, Z) + \frac{1}{2}T^c(X, Y, \psi_k Z) \\ &= -\frac{1}{2}\rho[F_2 \wedge \eta(X, \psi_k Y, Z) + F_2 \wedge \eta(X, Y, \psi_k Z)] \\ &= -\frac{1}{2}\rho[F_2(X, \psi_k Y)\eta(Z) + F_2(\psi_k Y, Z)\eta(X) + F_2(Y, \psi_k Z)\eta(X) + F_2(\psi_k Z, X)\eta(Y)].\end{aligned}$$

Since ψ_1 and ψ_2 commute we have $F_2(X, \psi_k Y) = g(X, \psi_2\psi_k Y) = g(X, \psi_k\psi_2 Y) = -g(\psi_k X, \psi_2 Y) = -F_2(\psi_k X, Y)$ and get

$$(\nabla^g_X F_k)(Y, Z) = -\frac{1}{2}\rho[F_2(X, \psi_k Y)\eta(Z) - F_2(X, \psi_k Z)\eta(Y)]. \tag{II.6}$$

The structure F_1 is of type $\mathcal{C}_7$ if $\delta F_1(\xi) = 0$ and

$$(\nabla^g_X F_1)(Y, Z) = \eta(Z)(\nabla^g_Y F_1)(X, \xi) - \eta(Y)(\nabla^g_{\psi_1 X} F_1)(\psi_1 Z, \xi).$$

The right side equals

$$\frac{1}{2}[\eta(Z)T^c(Y, \psi_1 X, \xi) - \eta(Y)T^c(\psi_1 X, \psi_1^2 Z, \xi)] = -\frac{1}{2}\rho[\eta(Z)F_2(Y, \psi_1 X) + \eta(Y)F_2(\psi_1 X, Z)]$$

and due to the calculation above, this proves the first statement.

With the equation (II.6) we have

$$(\nabla^g_X F_2)(Y, Z) = -\frac{1}{2}\rho(g(X, \psi_2^2 Y)\eta(Z) - g(X, \psi_2^2 Z)\eta(Y)) = -\frac{\delta F_2(\xi)}{4}(g(X, Y)\eta(Z) - g(X, Z)\eta(Y))$$

which proves that F_2 is of type $\mathcal{C}_6$. □

Example 3.22. Another example (see [Be12]) is given by the homogeneous space $M := \mathrm{SO}(3) \times \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ with the embedding

$$\mathrm{SO}(2) \ni A(t) := \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mapsto \left[A(t), A\left(\frac{t}{2}\right)^{-1}\right].$$

As an orthonormal basis of a reductive complement of $\mathfrak{so}(2)$ in $\mathfrak{so}(3) \times \mathfrak{sl}(2, \mathbb{R})$ we choose

$$e_1 := D_1\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, 0\right), \quad e_2 := D_1\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, 0\right), \quad e_3 := \frac{1}{2}D_2\left(0, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right),$$

$$e_4 := \frac{1}{2}D_2\left(0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right), \quad e_5 := \left(c_1\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, c_2\frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right),$$

such that $c_1 + c_2 \neq 0$, $D_1^2 = c_1(c_1 + c_2)$, $D_2^2 = -c_2(c_1 + c_2)$ and the numbers c_1, $-c_2$ and $(c_1 + c_2)$ have the same signature. We consider the almost contact structure (M, ξ, F) defined via

$$\xi := e_5 \text{ and } F = e_1 \wedge e_2 + e_3 \wedge e_4.$$

Then the characteristic connection ∇^c has torsion $T^c = -c_1 e_1 \wedge e_2 \wedge e_5 - c_2 e_3 \wedge e_4 \wedge e_5$.

Lemma 3.23. *The almost contact structure (M, ξ, F) is normal. Furthermore, the almost hermitian structure on $\bar{M}$, constructed with $a = \frac{-c_1 - c_2}{4}$, induced by the almost contact structure (M, ξ, F) is of class χ_3 and thus the structure (M, ξ, F) is of mixed class $\mathcal{C}_3 \oplus .. \oplus \mathcal{C}_8$.*

Proof. We use Theorem 3.11 and prove that the almost contact structure (M, ξ, F) is normal, satisfies $\delta F = (-c_1 - c_2)\eta$ and never satisfies $(\nabla^g_X F)(Y, Z) = a\eta(Y)g(X, Z) - a\eta(Z)g(X, Y)$ (Thus the structure on $\bar{M}$ is never Kähler and for $a = \frac{-c_1-c_2}{4}$ it really is of type χ_3).
First, the definition of F implies

$$\psi(e_1) = -e_2, \quad \psi(e_2) = e_1, \quad \psi(e_3) = -e_4, \quad \psi(e_4) = e_3, \quad \psi(e_5) = 0$$

and thus $e_1(\psi(X)) = e_2(X)$, $e_2(\psi(X)) = -e_1(X)$, $e_3(\psi(X)) = e_4(X)$, $e_4(\psi(X)) = -e_3(X)$ and we get

$$\begin{aligned} T^c(X, \psi Y, \psi Z) =& \eta(X)(-c_1 e_1 \wedge e_2(\psi Y, \psi Z) - c_2 e_3 \wedge e_4(\psi Y, \psi Z)) \\ =& \eta(X)(-c_1 e_1 \wedge e_2(Y, Z) - c_2 e_3 \wedge e_4(Y, Z)). \end{aligned}$$

This implies

$$T^c(X, Y, Z) = (-c_1 e_1 \wedge e_2 \wedge \eta - c_2 e_3 \wedge e_4 \wedge \eta)(X, Y, Z) = \mathfrak{S}_{X,Y,Z} T^c(X, \psi Y, \psi Z),$$

where $\mathfrak{S}$ denotes the cyclic sum. We have

$$\begin{aligned} N(X, Y, Z) =& (\nabla^g_X F)(Z, \psi Y) - (\nabla^g_Y F)(Z, \psi X) + (\nabla^g_{\psi X} F)(Z, Y) - (\nabla^g_{\psi Y} F)(Z, X) \\ &+ \eta(X)(\nabla^g_Y F)(\xi, \psi Z) - \eta(Y)(\nabla^g_X F)(\xi, \psi Z). \end{aligned}$$

Using the equality $(\nabla^g_X F)(Y, Z) = \frac{1}{2}(T^c(X, \psi Y, Z) + T^c(X, Y, \psi Z))$ we get

$$N(X, Y, Z) = T^c(X, Y, Z) - \mathfrak{S}_{X,Y,Z} T^c(X, \psi Y, \psi Z) = 0.$$

Secondly we calculate

$$\begin{aligned} \delta F(X) =& -\sum_i (\nabla^g_{e_i} F)(e_i, X) = -\frac{1}{2}\sum_i T^c(e_i, \psi e_i, X) \\ =& -\frac{1}{2}[-c_1 e_1 \wedge e_2 \wedge e_5(e_1, -e_2, X) - c_1 e_1 \wedge e_2 \wedge e_5(e_2, e_1, X) \\ &- c_2 e_3 \wedge e_4 \wedge e_5(e_3, -e_4, X) - c_2 e_3 \wedge e_4 \wedge e_5(e_4, e_3, X) \\ =& \eta(X)(-c_1 - c_2)]. \end{aligned}$$

Finally, to show that (M, ξ, F) never satisfies $(\nabla^g_X F)(Y, Z) = a\eta(Y)g(X, Z) - a\eta(Z)g(X, Y)$ we calculate

$$(\nabla^g_{e_1} F)(e_1, \xi) = \frac{c_1}{2} \text{ and } a\eta(e_1)g(e_1, \xi) - a\eta(\xi)g(e_1, e_1) = -a$$

as well as

$$(\nabla^g_{e_3} F)(e_3, \xi) = \frac{c_2}{2} \text{ and } a\eta(e_3)g(e_3, \xi) - a\eta(\xi)g(e_3, e_3) = -a.$$

Thus we can choose a such that the structure on $\bar{M}$ is integrable if $c_1 = c_2$. But in the construction of M we needed the condition that c_1 and $-c_2$ have the same signature and that $c_1 + c_2 \neq 0$, which contradicts $c_1 = c_2$. $\square$

In this example we only have Killing spinors with torsion satisfying $\nabla^s_X \phi = \alpha X \phi$ for $\alpha = 0$. But since the construction of $\bar{M}$ explicitly depends on $2\alpha = a \neq 0$, we cannot lift these spinors to $\bar{M}$.

3.4 Metric almost contact 3-structures

Let M be a manifold of dimension $n = 4m-1$ with 3 metric almost contact structures given by ξ_i, η_i and ψ_i for $i = 1, 2, 3$ (see [Ca09] for a more detailed description of metric almost contact 3-structures). Looking at the cone $\bar{M}$, we define the three almost hermitian structures

$$\begin{array}{lll} J_1(ar\partial_r) := \xi_1, & J_1(\xi_1) = -ar\partial_r, & J_1(V) = -\psi_1(V) \text{ for } V \perp \xi_1, \partial_r, \\ J_2(ar\partial_r) := \xi_2, & J_2(\xi_2) = -ar\partial_r, & J_2(V) = -\psi_2(V) \text{ for } V \perp \xi_2, \partial_r, \\ J_3(ar\partial_r) := -\xi_3, & J_3(\xi_3) = ar\partial_r, & J_3(V) = -\psi_3(V) \text{ for } V \perp \xi_3, \partial_r. \end{array}$$

Conversely, let $\bar{M}$ be a $4m$ dimensional manifold with three almost hermitian structures J_1, J_2 and J_3. We can define three almost contact structures

$$\begin{array}{rclrcl} \xi_1 & := & +aJ_1(\partial_r), & \psi_1(X) & := & -J_1(X) + \bar{g}(J_1(X), \partial_r)\partial_r. \\ \xi_2 & := & +aJ_2(\partial_r), & \psi_2(X) & := & -J_2(X) + \bar{g}(J_2(X), \partial_r)\partial_r. \\ \xi_3 & := & -aJ_3(\partial_r), & \psi_3(X) & := & +J_3(X) - \bar{g}(J_3(X), \partial_r)\partial_r. \end{array}$$

on $M = M \times \{1\} \subset \bar{M}$.
A $4m$-dimensional manifold with three almost hermitian structures $\omega_i = \bar{g}(., J_i.)$ is called *hyper-Kähler with torsion (HKT)* if the almost hermitian structures are integrable ($\bar{N}_i = 0$) and

$$J_1 \circ d\omega_1 = J_2 \circ d\omega_2 = J_3 \circ d\omega_3.$$

We can apply Theorem 3.2 to each of these structures and prove

Theorem 3.24. *The three almost hermitian structures on $\bar{M}$ satisfy the relation $J_1J_2 = -J_2J_1 = J_3$ if and only if ξ_1, ξ_2 and ξ_3 are orthonormal and the almost contact structures on M satisfy the following*

$$\begin{array}{rclrclrcl} \psi_3\psi_2 & = & -\psi_1 + \eta_2 \otimes \xi_3, & \psi_2\psi_3 & = & +\psi_1 + \eta_3 \otimes \xi_2, & \psi_1\psi_3 & = & -\psi_2 + \eta_3 \otimes \xi_1, \end{array} \tag{II.7}$$
$$\begin{array}{rclrclrcl} \psi_3\psi_1 & = & +\psi_2 + \eta_1 \otimes \xi_3, & \psi_2\psi_1 & = & -\psi_3 + \eta_1 \otimes \xi_2, & \psi_1\psi_2 & = & +\psi_3 + \eta_2 \otimes \xi_1, \end{array} \tag{II.8}$$

where η_i is the dual to ξ_i for $i = 1, 2, 3$. The appendant connection $\bar{\nabla}$ satisfies $\bar{\nabla}J_2 = \bar{\nabla}J_3 = \bar{\nabla}J_1 = 0$ if and only if the characteristic connections $\nabla^{c,i}$ on M of the three almost hermitian structures (η_i, ψ_i) are such that the corresponding connections ∇^i constructed in Definition 3.1 *coincide $\nabla^1 = \nabla^2 = \nabla^3 =: \nabla$.*
In this case, we get the additional commutator relations

$$[\xi_1, \xi_2] = 2a\xi_3 - T(\xi_1, \xi_2), \quad [\xi_2, \xi_3] = 2a\xi_1 - T(\xi_2, \xi_3), \quad [\xi_3, \xi_1] = 2a\xi_2 - T(\xi_3, \xi_1).$$

If furthermore the almost contact structures are normal, the three almost hermitian structures on $\bar{M}$ form an HKT structure.

Proof. Given three almost hermitian structures satisfying the relation $J_1J_2 = -J_2J_1 = J_3$, we compute

$$\begin{aligned} \psi_3(\psi_2(X)) &= -J_3(J_2(X)) + \bar{g}(J_3(J_2(X)), \partial_r)\partial_r + \bar{g}(J_2(X), \partial_r)J_3(\partial_r) - \bar{g}(J_3(X), \partial_r)\bar{g}(J_3(\partial_r), \partial_r)\partial_r \\ &= -\psi_1(X) - \bar{g}(X, J_2\partial_r)J_3(\partial_r) = -\psi_1(X) - a^2 g(X, J_2\partial_r)J_3(\partial_r) \\ &= -\psi_1(X) + g(X, \xi_2)\xi_3, \end{aligned}$$

and similarly for the other relations. Conversely, given three almost hermitian structures satisfying equations (II.7) and (II.8) we plug in ξ_1, ξ_2, and ξ_3 and, with $\psi_i(\xi_i) = 0$ for $i = 1, 2, 3$, we obtain immediately

$$\psi_1(\xi_2) = \xi_3, \quad \psi_1(\xi_3) = -\xi_2, \quad \psi_2(\xi_1) = -\xi_3, \quad \psi_2(\xi_3) = \xi_1, \quad \psi_3(\xi_1) = \xi_2, \quad \psi_3(\xi_2) = -\xi_1.$$

Since all ψ_i leave the vector space $V := \mathrm{span}(\xi_1, \xi_2, \xi_3)$ invariant and since they are orthonormal, they also leave $V^\perp$ invariant. For $X \perp \xi_1, \xi_2, \xi_3, \partial_r$ we have

$$J_1(J_2(X)) = \psi_1(\psi_2(X)) = \psi_3(X) = J_3(X) = -\psi_2(\psi_1(X)) = -J_2(J_1(X)).$$

For ξ_1 we obtain

$$J_1(J_2(\xi_1)) = -J_1(\psi_2(\xi_1)) = J_1(\xi_3) = -\psi_1(\xi_3) = \xi_2 = J_2(ar\partial_r) = -J_2(J_1(\xi_1)) = \psi_3(\xi_1) = J_3(\xi_1)$$

and similarly for ξ_2, ξ_3 and ∂_r. For a connection as in Theorem (3.2), we have that all almost hermitian structures are parallel under $\bar{\nabla}$ and for $X, Y \in TM$

$$[X, Y] = \bar{\nabla}^{\bar{g}}_X Y - \bar{\nabla}^{\bar{g}}_Y X = \bar{\nabla}_X Y - \bar{\nabla}_Y X - \bar{T}(X, Y).$$

Thus the commutator relations are given by

$$\begin{aligned}
[\xi_1, \xi_2] &= a^2[J_1(\partial_r), J_2(\partial_r)] = a^2(\bar{\nabla}_{J_1(\partial_r)} J_2(\partial_r) - \bar{\nabla}_{J_2(\partial_r)} J_1(\partial_r)) - \bar{T}(\xi_1, \xi_2) \\
&= a^2(J_2(\bar{\nabla}_{J_1(\partial_r)} \partial_r) - J_1(\bar{\nabla}_{J_2(\partial_r)} \partial_r)) - \bar{T}(\xi_1, \xi_2) \\
&= a^2(J_2(J_1(\partial_r)) - J_1(J_2(\partial_r))) - \bar{T}(\xi_1, \xi_2) \\
&= -2a^2 J_3(\partial_r) - \bar{T}(\xi_1, \xi_2) = 2a\xi_3 - T(\xi_1, \xi_2).
\end{aligned}$$

The other relations are to be calculated similarly.
If the almost contact structures are normal, then the almost hermitian structures are normal and with the formula for the torsion given in Remark 3.17 we have

$$\bar{g}(\bar{\nabla}_X, Y, Z) = \bar{g}(\bar{\nabla}^{\bar{g}}_X Y, Z) + \frac{1}{2}(J_i \circ d\omega_i)(X, Y, Z)$$

for any $i = 1, 2, 3$. This implies $J_1 \circ d\omega_1 = J_2 \circ d\omega_2 = J_3 \circ d\omega_3$. $\square$

Remarks 3.25.
- In [FFUV11, Section 7], the authors obtain a similar result (but without a description of the characteristic connections). Furthermore, they investigate more closely the conditions for the HKT structure to be *strong* ($J_i \circ d\omega_i$ is closed).

- If we rescale the metric such that $a = 1$ and if $T = 0$, we have 3 Kählerian structures on $\bar{M}$ and thus 3 Sasakian structures on M. Then the commutator relations in Theorem 3.24 ensure that the structures on M form a 3-Sasakian structure. This is Lemma 5 of [Bä93]: A one to one correspondence between hyper-Kähler structures on $\bar{M}$ and 3-Sasaki structures on M.

- We emphasize that it is not necessary that the three characteristic connections $\nabla^{c,i}$, $i = 1, 2, 3$ coincide in order to apply Theorem 3.24, only the connections ∇^i with torsion $T^i = T^{c,i} - 2a\eta_i \wedge F_i$ have to be equal. If M is a 3-Sasakian manifold, $T^i = 0$ for $i = 1, 2, 3$ and thus $\nabla^1 = \nabla^2 = \nabla^3 = \nabla^g$. In this case there exists a special G_2 structure on M which will be discussed in Example 4.18.

4 G_2 structures – Spin(7) structures on the cone

Let (M, g, Ψ, P) be a G_2 manifold (see Section 4 of Chapter I). For Spin(7) structures there is no one to one correspondence to spinors. Since we want to lift a G_2 structure to a Spin(7) structure on the cone, in this section it is more convenient to use the description of a G_2 structures via the differential form Ψ rather then via a spinor. Correspondences of geometric structures on non-twisted cones using the differential forms are considered in many other cases, see [II05] to name but one.
We cite a classical, but for us crucial result by Fernandez and Gray:

Lemma 4.1 ([FG82, Lemma 2.7])**.**

$$\begin{aligned}*\Psi(V,W,X,Y) &= g(P(V,W),P(X,Y)) - g(V,X)g(W,Y) + g(V,Y)g(W,X)\\ &= \Psi(V,W,P(X,Y)) - g(V,X)g(W,Y) + g(V,Y)g(W,X).\end{aligned}$$

Remark 4.2. In [FG82] this formula is stated differently,

$$\begin{aligned}*\Psi(V,W,X,Y) &= -g(P(V,W),P(X,Y)) + g(V,X)g(W,Y) - g(V,Y)g(W,X).\\ &= -\Psi(V,W,P(X,Y)) + g(V,X)g(W,Y) - g(V,Y)g(W,X).\end{aligned}$$

This is due to the standard 3-form Ψ used by Fernández and Gray, which corresponds to the orientation opposite to ours. This changes the sign of the Hodge operator.

Now we are able to prove

Lemma 4.3. *For any metric connection ∇ with skew torsion on M, the G_2 form Ψ satisfies*

$$(\nabla_Z * \Psi)(V,W,X,Y) = (\nabla_Z\Psi)(V,W,P(X,Y)) + (\nabla_Z\Psi)(X,Y,P(V,W)).$$

If ∇ satisfies $\nabla\Psi = a * \Psi$ for some $a > 0$, we have the simplified relation

$$\begin{aligned}(\nabla_Z * \Psi)(V,W,X,Y) &= a[\Psi(X,Y,V)g(Z,W) - \Psi(X,Y,W)g(Z,V)\\ &\quad +\Psi(V,W,X)g(Z,Y) - \Psi(V,W,Y)g(Z,X)].\end{aligned}$$

Proof. For any metric connection with skew torsion we have

$$\begin{aligned}(\nabla_Z * \Psi)(V,W,X,Y) =& Z * \Psi(V,W,X,Y) - *\Psi(\nabla_Z V,W,X,Y) - *\Psi(V,\nabla_Z W,X,Y)\\ &- *\Psi(V,W,\nabla_Z X,Y) - *\Psi(V,W,X,\nabla_Z Y).\end{aligned}$$

Since ∇ is metric, g is parallel and with Lemma 4.1 we get

$$\begin{aligned}=&Z\Psi(V,W,P(X,Y)) - \Psi(\nabla_Z V,W,P(X,Y)) - \Psi(V,\nabla_Z W,P(X,Y)) - \Psi(V,W,P(\nabla_Z X,Y))\\ &- \Psi(V,W,P(X,\nabla_Z Y)) - \Psi(V,W,\nabla_Z P(X,Y)) + \Psi(V,W,\nabla_Z P(X,Y)).\end{aligned}$$

We have $\Psi(V,W,(\nabla_Z P)(X,Y)) = g(P(V,W),(\nabla_Z P)(X,Y)) = (\nabla_Z\Psi)(X,Y,P(V,W))$ and thus we get

$$(\nabla_Z * \Psi)(V,W,X,Y) = (\nabla_Z\Psi)(V,W,P(X,Y)) + (\nabla_Z\Psi)(X,Y,P(V,W)).$$

The condition $\nabla\Psi = a * \Psi$ implies

$$(\nabla_Z * \Psi)(V,W,X,Y) = -a * \Psi(P(X,Y),Z,V,W) - a * \Psi(P(V,W),Z,X,Y)$$

and aplying once again Lemma 4.1 yields

$$\begin{aligned}&(\nabla_Z * \Psi)(V,W,X,Y) =\\ &= -a\Psi(P(X,Y),Z,P(V,W)) - a\Psi(P(V,W),Z,P(X,Y)) + ag(P(X,Y),V)g(Z,W)\\ &\quad -ag(P(X,Y),W)g(Z,V) + ag(P(V,W),X)g(Z,Y) - ag(P(V,W),Y)g(Z,X)\\ &= a[\Psi(X,Y,V)g(Z,W) - \Psi(X,Y,W)g(Z,V) + \Psi(V,W,X)g(Z,Y) - \Psi(V,W,Y)g(Z,X)],\end{aligned}$$

which finishes the proof. □

We define a 4-form on the cone $\bar{M}$ via

$$\Phi(\partial_r,X,Y,Z) := a^3r^3\Psi(X,Y,Z), \qquad \Phi(X,Y,Z,W) := a^4r^4 * \Psi(X,Y,Z,W)$$

for $X, Y, Z, W \in TM$. Since $\partial_r \lrcorner \Phi$ locally is a G_2-structure on $\partial_r^\perp$, Φ is a Spin(7)-structure on $\bar{M}$. As in Section 3, given a characteristic connection on M with respect to Ψ, we construct a connection ∇ with skew symmetric torsion T on M such that its lift $\bar{\nabla}$ to $\bar{M}$ with torsion $\bar{T}$ is the characteristic connection on $\bar{M}$ with respect to Φ. Since we have $T = \bar{T}_{|TM}$ and $\partial_r \lrcorner \bar{T} = 0$, we have $\bar{T} = T = 0$ in case of a parallel Spin(7) structure with respect to the Levi-Civita connection on $\bar{M}$, and thus ∇ is the Levi-Civita connection on M and thus T measures the difference of the G_2 structure to the nearly parallel case (see Remark 4.8).

Definition 4.4. Let (M, g, Ψ) be a G_2 manifold with characteristic connection ∇^c. We define a metric connection ∇ with skew symmetric torsion T via

$$T := T^c - \frac{2a}{3}\Psi.$$

As in the metric almost contact case (see the comments in Definition 3.1), T cannot be computed abstractly, but it is found through an educated guess and justified a posteriori from its properties.

Theorem 4.5. *The connection ∇ satisfies*

$$\nabla\Psi = a * \Psi,$$

and Φ is parallel with respect to $\bar{\nabla}$, the appendant connection on $\bar{M}$.

Proof. We have for the Riemannian connection ∇^g on M

$$\begin{aligned}
\nabla_X\Psi(Y,Z,W) &= X\Psi(Y,Z,W) - \Psi(\nabla^g_X Y, Z, W) - \Psi(Y, \nabla^g_X Z, W) - \Psi(Y, Z, \nabla^g_X W) \\
&\quad -\frac{1}{2}\Psi(T(X,Y),Z,W) - \frac{1}{2}\Psi(Y,T(X,Z),W) - \frac{1}{2}\Psi(Y,Z,T(X,W)) \\
&= (\nabla^c_X\Psi)(Y,Z,W) + \frac{1}{2}\Psi((T^c - T)(X,Y),Z,W) \\
&\quad +\frac{1}{2}\Psi(Y,(T^c-T)(X,Z),W) + \frac{1}{2}\Psi(Y,Z,(T^c-T)(X,W))
\end{aligned}$$

and because $\nabla^c\Psi = 0$ we have

$$\begin{aligned}
&\nabla_X\Psi(Y,Z,W) = \\
&= \frac{1}{2}[(T^c-T)(X,Y,P(Z,W)) + (T^c-T)(X,Z,P(W,Y)) + (T^c-T)(X,W,P(Y,Z))] \\
&= \frac{a}{3}[\Psi(X,Y,P(Z,W)) + \Psi(X,Z,P(W,Y)) + \Psi(X,W,P(Y,Z))].
\end{aligned}$$

With Lemma 4.1 we obtain

$$\begin{aligned}
a * \Psi(X,Y,Z,W) &= \frac{a}{3}[*\Psi(X,Y,Z,W) + *\Psi(X,Z,W,Y) + *\Psi(X,W,Y,Z)] \\
&= \frac{a}{3}[\Psi(X,Y,P(Z,W)) + \Psi(X,Z,P(W,Y)) + \Psi(X,W,P(Y,Z)) \\
&\quad -g(X,Z)g(Y,W) + g(X,W)g(Y,Z) - g(X,W)g(Z,Y) + g(X,Y)g(Z,W) \\
&\quad -g(X,Y)g(W,Z) + g(X,Z)g(W,Y)] \\
&= \nabla_X\Psi(Y,Z,W),
\end{aligned}$$

which proves the first statement. To show $\bar{\nabla}\Phi = 0$ on $\bar{M}$ we look at several cases. Let always be $V, W, X, Y, Z \in TM$.

<u>Case 1:</u> If ∂_r is one of the arguments, we compute

$$\begin{aligned}
(\bar{\nabla}_W\Phi)(\partial_r, X, Y, Z) &= Wa^3r^3\Psi(X,Y,Z) - \frac{1}{r}\Phi(W,X,Y,Z) - r^3a^3\Psi(\nabla_W X, Y, Z) \\
&\quad - r^3a^3\Psi(X,\nabla_W Y, Z) - r^3a^3\Psi(X,Y,\nabla_W Z)
\end{aligned}$$

$$= a^3r^3(\nabla_W\Psi)(X,Y,Z) - \frac{1}{r}\Phi(W,X,Y,Z) \;=\; a^4r^3 * \Psi(W,X,Y,Z) - \frac{1}{r}\Phi(W,X,Y,Z) \;=\; 0.$$

Case 2: If the direction of the derivative is equal to ∂_r, we obtain

$$\begin{aligned}(\bar\nabla_{\partial_r}\Phi)(X,Y,Z,W) &= \partial_r(a^4r^4 * \Psi(X,Y,Z,W)) - 4\frac{1}{r}\Phi(X,Y,Z,W)\\ &= 4r^3a^4 * \Psi(X,Y,Z,W) - 4\frac{1}{r}\Phi(X,Y,Z,W) \;=\; 0.\end{aligned}$$

Case 3: If the direction of the derivative and one argument are equal to ∂_r we compute

$$(\bar\nabla_{\partial_r}\Phi)(\partial_r,X,Y,Z) \;=\; \partial_r(a^3r^3\Psi(X,Y,Z)) - 3a^3r^3\frac{1}{r}\Psi(X,Y,Z) \;=\; 0.$$

Case 4: On TM we have:

$$\begin{aligned}&(\bar\nabla_V\Phi)(W,X,Y,Z) \;=\\ &= a^4r^4V * \Psi(W,X,Y,Z) - \Phi(\bar\nabla_V W,X,Y,Z) - \Phi(W,\bar\nabla_V X,Y,Z) - \Phi(W,X,\bar\nabla_V Y,Z)\\ &\quad -\Phi(W,X,Y,\bar\nabla_V Z)\\ &= a^4r^4V * \Psi(W,X,Y,Z) - \Phi(\nabla_V W - \frac{1}{r}\bar g(V,W)\partial_r,X,Y,Z) - \Phi(W,\nabla_V X - \frac{1}{r}\bar g(V,X)\partial_r,Y,Z)\\ &\quad -\Phi(W,X,\nabla_V Y - \frac{1}{r}\bar g(V,Y)\partial_r,Z) - \Phi(W,X,Y,\nabla_V Z - \frac{1}{r}\bar g(V,Z)\partial_r)\\ &= a^4r^4(\nabla_V * \Psi)(W,X,Y,Z) + r^4a^5[g(V,W)\Psi(X,Y,Z) - g(V,X)\Psi(W,Y,Z)\\ &\quad +g(V,Y)\Psi(W,X,Z) - g(V,Z)\Psi(W,X,Y)],\end{aligned}$$

which is equal to zero due to Lemma 4.3. □

Conversely, given a Spin(7) structure $(\bar M,\bar g,\Phi,\bar P,\bar p)$ on $\bar M$ (see Section 5 of Chapter I for the definitions), $\partial_r \lrcorner \Phi$ is a G_2 structure with respect to the metric a^2g on $M = M\times\{1\}\subset\bar M$ and thus

$$\Psi \;:=\; \frac{1}{a^3}\partial_r\lrcorner\Phi$$

defines a G_2 structure on M with respect to the metric g. To prove the following theorem, we need

Lemma 4.6. *If $*$ is the Hodge operator on M with respect to g and $*_{a^2g}$ is the Hodge operator on M with respect to the metric a^2g, we have for any 3-form ω*

$$*_{a^2g}\omega = a * \omega.$$

Proof. Let e_i for $i = 1..7$ be an orthonormal basis with dual basis e^i on M with respect to g. Then $\frac{1}{a}e_i$ with dual ae^i is a orthonormal basis with respect to a^2g. We define $e^{\{i,j,k\}} := e^i\wedge e^j\wedge e^k$ and $e^{\{i,j,k,j\}} := e^i\wedge e^j\wedge e^k\wedge e^l$ as well as $(se)^{\{i,j,k\}} := se^i\wedge se^j\wedge se^k$ for $s\in\mathbb{R}$ and $(se)^{\{i,j,k,j\}}$ respectively. Then we have

$$*_{a^2g}e^{\{i,j,k\}} = \frac{1}{a^3} *_{a^2g}(ae)^{\{i,j,k\}} = \frac{1}{a^3}(ae)^{\{1,..,7\}\setminus\{i,j,k\}} = \frac{1}{a^3}a^4e^{\{1,..,7\}\setminus\{i,j,k\}} = a * e^{\{i,j,k\}},$$

which proves the lemma. □

Theorem 4.7. *Given a* Spin(7) *structure on $\bar M$ with characteristic connection $\bar\nabla$ being the lift of a connection ∇ on M, we have for the G_2 structure Ψ induced by Φ*

$$\nabla\Psi = a * \Psi$$

and the characteristic connection on (M,g,Ψ) is given by $T^c = T + \frac{2a}{3}\Psi$.

Proof. We have for $W, X, Y, Z \in TM$

$$\begin{aligned}(\nabla_W \Psi)(X,Y,Z) &= \frac{1}{a^3}[W\Phi(\partial_r, X, Y, Z) \\ &- \Phi(\partial_r, \nabla_W X, Y, Z) - \Phi(\partial_r, X, \nabla_W Y, Z) - \Phi(\partial_r, X, Y, \nabla_W Z)] \\ &= \frac{1}{a^3}[(\bar\nabla_W \Phi)(\partial_r, X, Y, Z) + \Phi(\bar\nabla_W \partial_r, X, Y, Z)] = \frac{1}{a^3}\Phi(W, X, Y, Z).\end{aligned}$$

With Lemma 8 of [Bä93] and the definition of Ψ we conclude $\Phi|_{TM} = *_{a^2g}(\partial_r \lrcorner \Phi) = *_{a^2g}(a^3\Psi) = a^4 * \Psi$, where $*_{a^2g}$ is the Hodge operator on $M \subset \bar M$ with respect to the metric $a^2 g$. The last equality follows from Lemma 4.6. Thus we get

$$\nabla \Psi = a * \Psi.$$

For the connection ∇^c with torsion $T^c = T + \frac{2a}{3}\Psi$ we calculate as in the proof of Theorem 4.5

$$\begin{aligned}(\nabla^c_X \Psi)(Y,Z,W) &= (\nabla_X \Psi)(Y,Z,W) + \frac{1}{2}[(T-T^c)(X,Y,P(Z,W)) + (T-T^c)(X,Z,P(W,Y)) \\ &+ (T-T^c)(X,W,P(Y,Z))] \\ &= a * \Psi(X,Y,Z,W) - \frac{a}{3}[\Psi(X,Y,P(Z,W)) \\ &+ \Psi(X,Z,P(W,Y)) + \Psi(X,W,P(Y,Z))]\end{aligned}$$

which is equal to zero due to Lemma 4.1. Since the characteristic connection of a G_2 manifold is unique, this proves the theorem. □

Remark 4.8. In the case of an nearly parallel G_2 structure we have $T^c = \frac{2a}{3}\Psi$, i. e. $T = 0$ and thus $\nabla = \nabla^g$ lifts to the Levi-Civita connection on $\bar M$ and the corresponding Spin(7) structure on the cone is integrable. This means, that, as in the metric almost contact case, $T = T^c - \frac{2a}{3}\Psi$ measures the 'distance' of the G_2 structure from a nearly parallel G_2 structure.

4.1 The classification of G_2 structures and the corresponding classification of Spin(7) structures on the cone

We will now discuss the classification of Fernández [Fe86] of Spin(7) structures on $\bar M$ given in Section 5 of Chapter I, and compute the correspondence to the classification of G_2 structures [FG82]. Again we are only interested in structures carrying a characteristic connection (G_2 structures of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$). We write X_M for the projection on TM of a vector field X in $T\bar M$. We summarize some useful identities:

Lemma 4.9.

1. *P can be expressed through Ψ on TM:* $P(Y,Z) = \sum_l \Psi(e_l, Y, Z)e_l$.
2. *For any metric connection $\tilde\nabla$ with skew torsion on M, we have:*
$$\begin{aligned}(\tilde\nabla_X \Psi)(Y,Z,V) &= g((\tilde\nabla_X P)(Y,Z), V), \\ (\tilde\nabla_X P)(Y,Z) &= \sum_l g(e_l, (\tilde\nabla_X P)(Y,Z))e_l = \sum_l (\tilde\nabla_X \Psi)(e_l, Y, Z)e_l.\end{aligned}$$
3. *For ∇, this can be simplified to* $(\nabla_X P)(Y,Z) = a\sum_l *\Psi(X, e_l, Y, Z)e_l$.
4. *P, Ψ, and $\bar P$ are related by* $(X, Y, Z \in TM)$
$$\bar P(\partial_r, X, Y) \perp \partial_r, \quad \bar g(\bar P(X,Y,Z), \partial_r) = \Phi(X,Y,Z,\partial_r) = -a^3 r^3 \Psi(X,Y,Z),$$
$$\bar P(\partial_r, X, Y) = arP(X,Y), \quad \bar P(Y,Z,V)_M = ar^2(\nabla_Y P)(Z,V).$$

5. *The derivative of Φ on $\bar{M}$ can be expressed in terms of Ψ on M ($X, Y, Z, V, W \in TM$):*

$$(\bar{\nabla}^{\bar{g}}_X \Phi)(\partial_r, Z, V, W) = a^3 r^3 [(\nabla^g - \nabla)_X \Psi](Z, V, W),$$
$$(\bar{\nabla}^{\bar{g}}_X \Phi)(Y, Z, V, W) = a^4 r^4 [(\nabla^g - \nabla)_X * \Psi](Y, Z, V, W).$$

Proof. Statements (1)-(3) are easily checked. To prove statement (4) for $X, Y, Z \in TM$, we have

$$\bar{g}(\bar{P}(\partial_r, X, Y), Z) = \Phi(\partial_r, X, Y, Z) = a^3 r^3 \Psi(X, Y, Z) = ar\bar{g}(P(X, Y), Z),$$

thus $\bar{P}(\partial_r, X, Y) = arP(X, Y)$. Furthermore,

$$\begin{aligned} \bar{g}(X, \bar{P}(Y, Z, V)) &= \Phi(Y, Z, V, X) \;=\; a^3 r^4 (\nabla_Y \Psi)(Z, V, X) \;=\; a^3 r^4 g(X, (\nabla_Y P)(Z, V)) \\ &= ar^2 \bar{g}(X, (\nabla_Y P)(Z, V)), \end{aligned}$$

and thus $\bar{P}(Y, Y, V)_M = ar^2 (\nabla_Y P)(Z, V)$. For (5) and vector fields $X, Y, Z, V, W \in TM$, we calculate

$$\begin{aligned} &2(\bar{\nabla}^{\bar{g}}_X \Phi)(\partial_r, Z, V, W) \;= \\ &= \; 2(\bar{\nabla}_X \Phi)(\partial_r, Z, V, W) + \Phi(\partial_r, \bar{T}(X, Z), V, W) + \Phi(\partial_r, Z, \bar{T}(X, V), W) + \Phi(\partial_r, Z, V, \bar{T}(X, W)) \\ &= \; a^3 r^3 [\Psi(T(X, Z), V, W) + \Psi(Z, T(X, V), W) + \Psi(Z, V, T(X, W))] \\ &= \; 2a^3 r^3 [\Psi((\nabla_X - \nabla^g_X) Z, V, W) + \Psi(Z, (\nabla_X - \nabla^g_X) V, W) + \Psi(Z, V, (\nabla_X - \nabla^g_X) W)] \\ &= \; -2a^3 r^3 [(\nabla - \nabla^g)_X \Psi](Z, V, W), \end{aligned}$$

and similarly

$$\begin{aligned} &(\bar{\nabla}^{\bar{g}}_X \Phi)(Y, Z, V, W) \;= \\ &= \; \frac{1}{2}[\Phi(\bar{T}(X, Y), Z, V, W) + \Phi(Y, \bar{T}(X, Z), V, W) + \Phi(Y, Z, \bar{T}(X, V), W) + \Phi(Y, Z, V, \bar{T}(X, W))] \\ &= \; \frac{a^4 r^4}{2}[*\Psi(T(X, Y), Z, V, W) + *\Psi(Y, T(X, Z), V, W) \\ &\quad\; + *\Psi(Y, Z, T(X, V), W) + *\Psi(Y, Z, V, T(X, W))] \\ &= \; -a^4 r^4 [(\nabla - \nabla^g)_X * \Psi](Y, Z, V, W) = a^4 r^4 [(\nabla^g - \nabla)_X * \Psi](Y, Z, V, W), \end{aligned}$$

which finishes the proof. □

Remark 4.10. Since the characteristic connection of the Spin(7) structure on $\bar{M}$ is unique (see Section 5 of Chapter I), we can conclude for any such structure satisfying $\bar{\nabla}^{\bar{g}} \Phi = 0$ that $\nabla = \nabla^g$ and thus $\nabla^g \Psi = a * \Psi$ and the G_2 structure is of class $\mathcal{W}_1$. Conversely, given a connection ∇ with skew symmetric torsion and $\nabla \Psi = a * \Psi$ we construct ∇^c via $T^c := T - \frac{2a}{3}\Psi$, which satisfies $\nabla^c \Psi = 0$ and thus is unique. Hence a metric connection with skew symmetric torsion and the property $\nabla \Psi = *\Psi$ is unique.

For any tensor R on M we introduce the notation $R \llcorner X$ to denote $R(-, X)$.
We extend the metric g to arbitrary k-tensors R, S via an orthonormal frame $e_1, \ldots, e_n$

$$g(R, S) \;:=\; \sum_{i_1, .., i_k = 1}^{n} R(e_{i_1}, .., e_{i_k}) S(e_{i_1}, .., e_{i_k}).$$

Lemma 4.11. *A* Spin(7) *structure on $\bar{M}$ is of class $\mathcal{U}_1$ if and only if on M*

- *$g(\nabla^g \Psi, *\Psi) = ag(*\Psi, *\Psi)$, and*
- *for every $X \in TM$ we have $g(*\Psi, [(\nabla - \nabla^g) * \Psi] \llcorner X) = 3g(\Psi, [(\nabla - \nabla^g)\Psi] \llcorner X)$.*

The structure on $\bar{M}$ is of class $\mathcal{U}_2$ if and only if the following conditions are satisfied for $X, Y, Z, X_1, .., X_4 \in TM$ and a local orthonormal frame $e_1, .., e_7$ of TM:

- $\delta\Phi|_{TM} = 0$ *on TM, which is equivalent to* $0 = \sum_{i=1}^{7}[(\nabla^g - \nabla)_{e_i} * \Psi](e_i, X, Y, Z)$

- $0 = \sum_{i=1}^{4} \sum_{l<j<8} (-1)^i \delta\Psi(e_l, e_j)\Psi(e_l, e_j, X_i)\Psi(X_1, .., \hat{X}_i, .., X_4)$

- $28[(\nabla^g - \nabla)_W * \Psi](X_1, X_2, X_3, X_4)$
  $= \sum_{i=1}^{4} \sum_{l<j<8} (-1)^{i+1} \delta\Psi(e_l, e_j)\Psi(e_l, e_j, X_i) * \Psi(W, X_1, .., \hat{X}_i, .., X_4).$

Proof. We consider a local $\bar{g}$-orthonormal frame $\bar{e}_1 = \frac{1}{ar}e_1, .., \bar{e}_7 = \frac{1}{ar}e_7, e_8 = \partial_r$ of $T\bar{M}$ such that $e_1, .., e_7$ is a local orthonormal frame of TM. With Lemma 4.2 of [Fe86] a Spin(7) structure is defined to be of class $\mathcal{U}_1$ if and only if

$$0 \;=\; -6\delta\Phi(\bar{p}(X)) \;=\; \sum_{i,k,j=1}^{8} (\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_j, \bar{e}_k, \bar{P}(\bar{e}_i, \bar{e}_j, \bar{e}_k), X).$$

For $X \in TM$ we have

$$\begin{aligned}
0 \;&=\; -6\delta\Phi(\bar{p}(X)) \;=\; \sum_{i,k,j=1}^{8} (\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_j, \bar{e}_k, \bar{P}(\bar{e}_i, \bar{e}_j, \bar{e}_k), X) \\
&= \sum_{i,k,j=1}^{7} (\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_j, \bar{e}_k, \bar{P}(\bar{e}_i, \bar{e}_j, \bar{e}_k), X) + 2\sum_{i,j=1}^{7} (\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_j, \partial_r, \bar{P}(\bar{e}_i, \bar{e}_j, \partial_r), X) \\
&= \frac{1}{a^6r^6}\sum_{i,k,j=1}^{7} (\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, e_k, ar^2(\nabla_{e_i}P)(e_j, e_k) + \bar{g}(\bar{P}(e_i, e_j, e_k), \partial_r)\partial_r, X) \\
&\quad +2\frac{1}{a^4r^4}\sum_{i,j=1}^{7} (\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, \partial_r, arP(e_i, e_j), X) \\
&= \frac{1}{a^5r^4}\sum_{i,k,j=1}^{7} a^4r^4[(\nabla^g - \nabla)_{e_i} * \Psi](e_j, e_k, (\nabla_{e_i}P)(e_j, e_k), X) \\
&\quad -\frac{1}{a^3r^3}\sum_{i,k,j=1}^{7} \Psi(e_i, e_j, e_k)(\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, e_k, \partial_r, X) - 2\frac{a^3r^3}{a^3r^3}\sum_{i,j=1}^{7} ([\nabla^g - \nabla]_{e_i}\Psi)(e_j, P(e_i, e_j), X) \\
&= \sum_{i,k,j,l=1}^{7} [(\nabla^g - \nabla)_{e_i} * \Psi](e_j, e_k, *\Psi(e_i, e_l, e_j, e_k)e_l, X) \\
&\quad -3\sum_{i,k,j=1}^{7} \Psi(e_i, e_j, e_k)([\nabla^g - \nabla]_{e_i}\Psi)(e_j, e_k, X) \\
&= g(*\Psi, (\nabla^g - \nabla) * \Psi \llcorner X) - 3g(\Psi, (\nabla^g - \nabla)\Psi \llcorner X).
\end{aligned}$$

In case $X = \partial_r$, we deduce from Lemma 4.9:

$$\begin{aligned}
0 =& \sum_{i,j,k=1}^{7} (\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, e_k, \bar{P}(e_i, e_j, e_k), \partial_r) \;=\; ar^2 \sum_{i,j,k=1}^{7} (\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, e_k, (\nabla_{e_i}P)(e_j, e_k), \partial_r) \\
=& a^4 r^5 \sum_{i,j,k,l=1}^{7} [(\nabla^g - \nabla)_{e_i}\Psi](e_j, e_k, e_l) g((\nabla_{e_i}P)(e_j, e_k), e_l) \\
=& -a^4 r^5 [\sum_{i,j,k,l=1}^{7} (\nabla^g_{e_i}\Psi)(e_j, e_k, e_l)(\nabla_{e_i}\Psi)(e_j, e_k, e_l) - \sum_{i,j,k,l=1}^{7} (\nabla_{e_i}\Psi)(e_j, e_k, e_l)(\nabla_{e_i}\Psi)(e_j, e_k, e_l)] \\
=& -a^4 r^5 [g(\nabla^g\Psi, \nabla\Psi) - g(\nabla\Psi, \nabla\Psi)] = -a^5 r^5 [g(\nabla^g\Psi, *\Psi) - ag(*\Psi, *\Psi)],
\end{aligned}$$

and thus we have $g(\nabla^g\Psi, *\Psi) = ag(*\Psi, *\Psi)$. A Spin(7) structure is of class $\mathcal{U}_2$ if it satisfies

$$\begin{aligned}
28(\bar{\nabla}^{\bar{g}}_W\Phi)(X_1, X_2, X_3, X_4) = -\sum_{i=1}^{4}(-1)^{i+1}[&\delta\Phi(\bar{p}(X_i))\Phi(W, X_1, .., \hat{X}_i, .., X_4) \\
&+ 7\bar{g}(W, X_i)\delta\Phi(X_1, .., \hat{X}_i, .., X_4)].
\end{aligned} \tag{II.9}$$

Suppose $W = X_1 = \partial_r$ and $X_2, X_3, X_4 \in TM$. For a 3-form ξ on TM we have

$$\bar{g}(\bar{p}(\partial_r), \xi) = \bar{g}(\partial_r, \bar{P}(\xi)) = -\Phi(\partial_r, \xi) = -a^3 r^3 \Psi(\xi) = \bar{g}(-a^3 r^3 \Psi, \xi)$$

and thus $\bar{p}(\partial_r) = -a^3 r^3 \Psi$. Since $\partial_r \lrcorner \bar{T} = 0$ we have $\bar{\nabla}^{\bar{g}}_{\partial_r}\Phi = 0$ and the defining relation of the class $\mathcal{U}_2$ reduces to

$$0 = \delta\Phi(p(\partial_r))\Phi(\partial_r, X_2, X_3, X_4) + 7\delta\Phi(X_2, X_3, X_4) = \delta\Phi(-a^6 r^6 \Psi(X_2, X_3, X_4)\Psi + 7X_2 \wedge X_3 \wedge X_4).$$

Since $a^6 r^6 \Psi(X_2, X_3, X_4)\Psi - 7X_2 \wedge X_3 \wedge X_4$ spans $\Lambda^3(TM)$ we have $\delta\Phi = 0$ on TM. For $X, Y, Z \in TM$ we have

$$\begin{aligned}
0 = \delta\Phi(X, Y, Z) = -\sum_{i=1}^{8}(\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_i, X, Y, Z) = -\frac{1}{a^2 r^2}\sum_{i=1}^{7}(\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_i, X, Y, Z) \\
= -a^2 r^2 \sum_{i=1}^{7}[(\nabla^g - \nabla)_{e_i} * \Psi](e_i, X, Y, Z).
\end{aligned}$$

For $X \in TM$ we have

$$\begin{aligned}
\delta\Phi(\bar{p}(X)) =&\delta\Phi(\sum_{i<j<k=1}^{8} \bar{g}(\bar{p}(X), \bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_k)\bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_k) \\
=&\delta\Phi(\sum_{i<j<8} \bar{g}(\bar{p}(X), \bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_8)\bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_8) \\
=&\sum_{i<j<8} \bar{g}(\bar{p}(X), \bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_8)\delta\Phi(\bar{e}_i, \bar{e}_j, \partial_r) \\
=&-\sum_{k=1}^{7}\sum_{i<j<8} (\bar{\nabla}^{\bar{g}}_{\bar{e}_k}\Phi)(\bar{e}_k, \bar{e}_i, \bar{e}_j, \partial_r)\bar{g}(X, \bar{P}(\bar{e}_i, \bar{e}_j, \partial_r)) \\
=&\sum_{k=1}^{7}\sum_{i<j<8} a^3r^3(\nabla^g_{\bar{e}_k}\Psi)(\bar{e}_k, \bar{e}_i, \bar{e}_j)\Phi(\bar{e}_i, \bar{e}_j, \partial_r, X) \\
=&a^6r^6\sum_{k=1}^{7}\sum_{i<j<8} (\nabla^g_{\bar{e}_k}\Psi)(\bar{e}_k, \bar{e}_i, \bar{e}_j)\Psi(\bar{e}_i, \bar{e}_j, X) \\
=&-\sum_{i<j<8} \delta\Psi(e_i, e_j)\Psi(e_i, e_j, X).
\end{aligned}$$

Suppose $W = \partial_r$ and $X_1, .., X_4 \in TM$. Then equation (II.9) gives us

$$\begin{aligned}
0 &= \sum_{i=1}^{4}(-1)^{i+1}\delta\Phi(\bar{p}(X_i))a^3r^3\Psi(X_1, .., \hat{X}_i, .., X_4) \\
&= a^3r^3\sum_{i=1}^{4}\sum_{l<j<8} (-1)^i\delta\Psi(e_l, e_j)\Psi(e_l, e_j, X_i)\Psi(X_1, .., \hat{X}_i, .., X_4).
\end{aligned}$$

For $W, X_i \in TM$, equation (II.9) reduces to

$$28(\bar{\nabla}^{\bar{g}}_W\Phi)(X_1, X_2, X_3, X_4) = 28a^4r^4[(\nabla^g - \nabla)_W * \Psi](X_1, X_2, X_3, X_4),$$

which is equal to

$$\begin{aligned}
&-\sum_{i=1}^{4}(-1)^{i+1}[\delta\Phi(\bar{p}(X_i))\Phi(W, X_1, .., \hat{X}_i, .., X_4) + 7\bar{g}(W, X_i)\delta\Phi(X_1, .., \hat{X}_i, .., X_4)] \\
&= a^4r^4\sum_{i=1}^{4}\sum_{l<j<8} (-1)^{i+1}\delta\Psi(e_l, e_j)\Psi(e_l, e_j, X_i) * \Psi(W, X_1, .., \hat{X}_i, .., X_4).
\end{aligned}$$

This proves the statement. □

Remark 4.12. One can use Lemma 4.3 and Lemma 4.9 to simplify these equations in rather lengthly calculations. The property

$$0 = \sum_{i=1}^{7}[(\nabla^g - \nabla)_{e_i} * \Psi](e_i, X, Y, Z)$$

can for example be simplified to

$$0 = g((\Psi\llcorner Y)\llcorner Z, \delta\Psi\llcorner X) + g(\Psi\llcorner X, (\nabla^g\Psi\llcorner Y)\llcorner Z) - g(\Psi\llcorner X, (*\Psi\llcorner Y)\llcorner Z).$$

Another simplification (see Lemma 4.17) will be used in Example 4.18.

Theorem 4.13. *If the* Spin(7) *structure on the cone* $\bar{M}$ *is of class* $\mathcal{U}_1$*, then:*

- *The* G_2 *structure* Ψ *on* M *cannot be of class* $\mathcal{W}_3 \oplus \mathcal{W}_4$.
- *The* G_2 *structure is of class* $\mathcal{W}_1$ *if and only if the* Spin(7) *structure is integrable.*

If the structure on $\bar{M}$ *is of class* $\mathcal{U}_2$*, then the structure on* M *is never of class* $\mathcal{W}_1 \oplus \mathcal{W}_3$.

Proof. Since the relation $g(\nabla^g\Psi, *\Psi) = 0$ defines the class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, we conclude the first result directly from Lemma 4.11. Now, assume the G_2 structure Ψ is of class $\mathcal{W}_1$, i.e. nearly parallel G_2 (see [FG82]):

$$\nabla^g\Psi = \frac{1}{168} g(\nabla^g\Psi, *\Psi) * \Psi.$$

Taking the scalar product with $*\Psi$ on both sides leads to

$$g(\nabla^g\Psi, *\Psi) = \frac{1}{168} g(\nabla^g\Psi, *\Psi) g(*\Psi, *\Psi).$$

With the Spin(7) structure being of class $\mathcal{U}_1$ and the calculation above we get $g(*\Psi, *\Psi) = \frac{1}{168} g(*\Psi, *\Psi) g(*\Psi, *\Psi)$ and thus $g(*\Psi, *\Psi) = 168$. Therefore,

$$\nabla^g\Psi = \frac{1}{168} g(\nabla^g\Psi, *\Psi) * \Psi = a\frac{1}{168} g(*\Psi, *\Psi) * \Psi = a * \Psi.$$

Thus $\nabla^g\Psi = \nabla\Psi = a * \Psi$ and with Remark 4.10 we get $\nabla = \nabla^g$ and $\bar{\nabla}^{\bar{g}} = \bar{\nabla}$. Since $\bar{\nabla}\Phi = 0$ the Spin(7) structure on $\bar{M}$ is integrable.
Consider a structure on $\bar{M}$ of class $\mathcal{U}_2$. With Lemma 4.11 we get $\delta\Phi = 0$ on TM. To see that this structure is integrable it is sufficient to show $\partial_r \lrcorner \delta\Phi = 0$, see [Fe86]. We have for $X, Y \in TM$

$$\begin{aligned}(\partial_r \lrcorner \delta\Phi)(X,Y) &= -\sum_{i=1}^{8} (\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_i, \partial_r, X, Y) = ar\sum_{i=1}^{7}((\nabla^g - \nabla)_{e_i}\Psi)(e_i, X, Y) \\ &= ar\sum_{i=1}^{7}[(\nabla^g_{e_i}\Psi)(e_i, X, Y) + \Phi(e_i, e_i, X, Y)] = -ar\delta\Psi(X,Y).\end{aligned}$$

This is equal to zero if the structure on M is cocalibrated (of class $\chi_1 \oplus \chi_3$, defined by $\delta\Psi = 0$). □

4.2 Corresponding spinors on G_2 manifolds and their cones

Since we have $T - T^c = -\frac{2a}{3}\Psi$, the difference $\bar{T} - \overline{T^c}$ is the lift of $a^2r^2T - a^2r^2T^c = -\frac{2a}{3}a^2r^2\Psi$. Furthermore, $\frac{1}{a^3r^3}\partial_r \lrcorner \Phi$ is the lift of Ψ to $\bar{M}$, hence we have

$$\bar{T} - \overline{T^c} = -\frac{2}{3r}\partial_r \lrcorner \Phi.$$

Now Lemma 2.4 implies:

Theorem 4.14. *For a* G_2 *manifold with characteristic connection* ∇^c *and for* $\alpha = \frac{1}{2}a$ *or* $\alpha = -\frac{1}{2}a$*, there is*

1. *a one to one correspondence between Killing spinors with torsion*

$$\nabla^s_X\phi = \alpha X\phi$$

on M*, and parallel spinors of the connection* $\bar{\nabla}^s + \frac{4s}{3r}\partial_r \lrcorner \Phi$ *on* $\bar{M}$ *with cone constant* a

$$\bar{\nabla}^s_X\phi + \frac{2s}{3r}(X \lrcorner (\partial_r \lrcorner \Phi))\phi = 0.$$

2. *a one to one correspondence between $\bar{\nabla}^s$-parallel spinors on $\bar{M}$ with cone constant a and spinors on M satisfying*
$$\nabla^s_X \phi = \alpha X\phi + \frac{2as}{3}(X \lrcorner \Psi)\phi.$$

In particular for $s = \frac{1}{4}$ we get the correspondence

spinors on M	*spinors on $\bar{M}$*
$\nabla^c_X \phi = \alpha X \phi$	$\bar{\nabla}_X \phi = -\frac{1}{6r}(X \lrcorner (\partial_r \lrcorner \Phi))\phi$
$\nabla^c_X \phi = \alpha X\phi + \frac{a}{6}(X \lrcorner \Psi)\ \phi$	$\bar{\nabla}_X \phi = 0$

Remark 4.15. As for metric almost contact structures (see Remark 3.17), one can use the characterisation $\bar{T} = -\delta\Phi - \frac{7}{6} * (\theta \wedge \Phi)$ with $\theta = \frac{1}{7} * (\delta\Phi \wedge \Phi)$ (see [Iv04]) and the description of T^c given in Theorem 4.8 of [FI02] to rewrite these equations in terms of the geometric data of the Spin(7) structure.

Remark 4.16. In Section 3 of Chapter I we see, that a G_2 structure also is given by a spinor ϕ, which is ∇^c-parallel. On the other hand, for any Spin(7) manifold there is a spinor that is parallel with respect to the characteristic connection (Theorem 1.1 of [Iv04]). The G_2 spinor induces the Spin(7) spinor in the following way:
From Lemma 4.2 we know that $(X \lrcorner \Psi) \cdot \phi = -3X \cdot \phi$ and thus
$$\nabla^c_X \phi = 0 = 3X \cdot \phi + (X \lrcorner \Psi) \cdot \phi,$$
which is the identity for a spinor on M inducing a $\bar{\nabla}$-parallel spinor on $\bar{M}$ in Theorem 4.14. Be cautious that ∇^c may have more parallel spinor fields than just ϕ; for these, we cannot define a suitable 'lifted' spinor on the cone, unless one finds a similar trick to write the spinor field equation in a form covered by Theorem 4.14.

This remark indicates, that one might use the description of a Spin(7) and a G_2 structure via spinors to calculate the correspondences given in Theorem 4.13 as it was done in Section 1 for SU(3) and G_2 manifolds. Note that the stabilizer of a spinor in dimension 8 is not always the group Spin(7), so one does not get correspondences as in Lemmas 3.1 and 4.1 of Chapter I.

4.3 Examples

To simplify the calculations in the example we reformulate the second condition for a G_2 structure on M to imply a Spin(7) structure of class $\mathcal{U}_1$ on $\bar{M}$ of Lemma 4.11. So we only have to calculate Ψ, $*\Psi$ and $\nabla^g\Psi$ to check the conditions.

Lemma 4.17. *The second condition of Lemma* 4.11
$$g(*\Psi, [(\nabla - \nabla^g) * \Psi]\llcorner X) \;=\; 3g(\Psi, [(\nabla - \nabla^g)\Psi]\llcorner X)$$
is equivalent to

$$\begin{aligned}
0 = \sum_{i,k,j,l,m=1}^{7} \Big[& * \Psi(e_i, e_j, e_k, e_l)(\nabla^g_{e_i}\Psi)(e_j, e_k, e_m)\Psi(e_m, e_l, X) \\
& + *\Psi(e_i, e_j, e_k, e_l)(\nabla^g_{e_i}\Psi)(e_l, X, e_m)\Psi(e_m, e_j, e_k) - *\Psi(e_i, e_j, e_k, e_l) * \Psi(e_i, e_j, e_k, e_m)\Psi(e_m, e_l, X) \\
& - *\Psi(e_i, e_l, e_j, e_k) * \Psi(e_i, e_l, X, e_m)\Psi(e_m, e_j, e_k)\Big] \\
& + 3 \sum_{i,k,j=1}^{7} \Big[- \Psi(e_i, e_j, e_k)(\nabla^g_{e_i}\Psi)(e_j, e_k, X) + a\,\Psi(e_i, e_j, e_k) * \Psi(e_i, e_j, e_k, X)\Big].
\end{aligned}$$

Proof. We continue the calculation of the proof of Lemma 4.11 and with Lemma 4.3 we get

$$
\begin{aligned}
0 =& \frac{1}{a}\sum_{i,k,j,l=1}^{7}(\nabla^g_{e_i} * \Psi)(e_j, e_k, a * \Psi(e_i, e_l, e_j, e_k)e_l, X) \\
&- \frac{1}{a}\sum_{i,k,j,l=1}^{7}(\nabla_{e_i} * \Psi)(e_j, e_k, a * \Psi(e_i, e_l, e_j, e_k)e_l, X) \\
&- 3\sum_{i,k,j=1}^{7}\Psi(e_i, e_j, e_k)(\nabla^g_{e_i}\Psi)(e_j, e_k, X) + 3a\sum_{i,k,j=1}^{7}\Psi(e_i, e_j, e_k) * \Psi(e_i, e_j, e_k, X) \\
=& \sum_{i,k,j,l=1}^{7} *\Psi(e_i, e_l, e_j, e_k)(\nabla^g_{e_i} * \Psi)(e_j, e_k, e_l, X) \\
&- \sum_{i,k,j,l=1}^{7} *\Psi(e_i, e_l, e_j, e_k)(\nabla_{e_i} * \Psi)(e_j, e_k, e_l, X) \\
&- 3\sum_{i,k,j=1}^{7}\Psi(e_i, e_j, e_k)(\nabla^g_{e_i}\Psi)(e_j, e_k, X) + 3a\sum_{i,k,j=1}^{7}\Psi(e_i, e_j, e_k) * \Psi(e_i, e_j, e_k, X)
\end{aligned}
$$

$$
\begin{aligned}
=& -\sum_{i,k,j,l=1}^{7} *\Psi(e_i, e_l, e_j, e_k)(\nabla^g_{e_i}\Psi)(e_j, e_k, P(e_l, X)) \\
&- \sum_{i,k,j,l=1}^{7} *\Psi(e_i, e_l, e_j, e_k)(\nabla^g_{e_i}\Psi)(e_l, X, P(e_j, e_k)) \\
&+ \sum_{i,k,j,l=1}^{7} *\Psi(e_i, e_l, e_j, e_k)(\nabla_{e_i}\Psi)(e_j, e_k, P(e_l, X)) \\
&+ \sum_{i,k,j,l=1}^{7} *\Psi(e_i, e_l, e_j, e_k)(\nabla_{e_i}\Psi)(e_l, X, P(e_j, e_k)) \\
&- 3\sum_{i,k,j=1}^{7}\Psi(e_i, e_j, e_k)(\nabla^g_{e_i}\Psi)(e_j, e_k, X) + 3a\sum_{i,k,j=1}^{7}\Psi(e_i, e_j, e_k) * \Psi(e_i, e_j, e_k, X)
\end{aligned}
$$

which is equal to the condition stated in the lemma. □

Example 4.18. Let $(M, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ be a 7 dimensional 3-Sasaki manifold with corresponding 2-forms F_i, $i = 1, 2, 3$. Let η_i for $i = 1, .., 7$ be the dual of a local basis $\{e_1 = \xi_1, e_2 = \xi_2, e_3 = \xi_3, e_4, .., e_7\}$, such that

$$F_1 = -\eta_{23} - \eta_{45} - \eta_{67}, \quad F_2 = \eta_{13} - \eta_{46} + \eta_{57}, \quad F_3 = -\eta_{13} - \eta_{47} - \eta_{56}.$$

Here for $\eta_i \wedge .. \wedge \eta_j$ we write $\eta_{i,..,j}$. In [AF10] it is explained that there is no characteristic connection as such, but one can construct a cocalibrated G_2 structure

$$\Psi = \eta_1 \wedge F_1 + \eta_2 \wedge F_2 + \eta_3 \wedge F_3 + 4\eta_1 \wedge \eta_2 \wedge \eta_3 = \eta_{123} - \eta_{145} - \eta_{167} - \eta_{246} + \eta_{257} - \eta_{347} - \eta_{356}$$

with characteristic connection ∇^c and torsion $T^c = \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3$ that is very well adapted to the 3-Sasakian structure. It is therefore called the *canonical G_2 structure* of the

underlying 3-Sasakian structure. Remark 4.16 ensures then the existence of a $\bar{\nabla}$-parallel spinor field on $\bar{M}$.
We calculate the class of the Spin(7) structure on $\bar{M}$ of the canonical G_2 structure using Lemma 4.11.

Theorem 4.19. *The* Spin(7) *structure on the cone constructed from the canonical* G_2 *structure of a* 3*-Sasakian manifold is of class* $\mathcal{U}_1$ *if and only if the cone constant is* $a = \frac{15}{14}$*.*

Proof. Due to the formulation of the second condition of Lemma 4.11 given in Lemma 4.17, we just need to calculate $*\Psi$ and $\nabla^g\Psi$. Obviously $*\Psi$ is given by

$$*\Psi = \eta_{4567} - \eta_{2367} - \eta_{2345} - \eta_{1357} + \eta_{1346} - \eta_{1256} - \eta_{1247}.$$

To get $\nabla^g\Psi$ we observe

$$\begin{aligned}\nabla^g_{e_j}\Psi = {} & (\nabla^g_{e_j}\eta_1)\wedge F_1 + (\nabla^g_{e_j}\eta_2)\wedge F_2 + (\nabla^g_{e_j}\eta_3)\wedge F_3 \\ & + \eta_1 \wedge (\nabla^g_{e_j}F_1) + \eta_2 \wedge (\nabla^g_{e_j}F_2) + \eta_3 \wedge (\nabla^g_{e_j}F_3) \\ & + 4(\nabla^g_{e_j}\eta_1)\wedge\eta_2\wedge\eta_3 + 4\eta_1\wedge(\nabla^g_{e_j}\eta_2)\wedge\eta_3 + 4\eta_1\wedge\eta_2\wedge(\nabla^g_{e_j}\eta_3)\end{aligned}$$

and since (η_i, F_i) are Sasakian structures we have $(\nabla^g_{e_j}F_i)(Y,Z) = g(e_j,Z)\eta_i(Y) - g(e_j,Y)\eta_i(Z)$. Thus $\nabla^g_{e_j}F_i = \eta_j \wedge \eta_i$ for $i = 1,2,3$ and $j = 1,..,7$ implying $\eta_i \wedge (\nabla^g_{e_j}F_i) = 0$. Since

$$(\nabla^g_X\eta_i)Y = g(Y, \nabla^g_X\xi_i) = g(Y, -\Psi_i X) = F_i(X,Y)$$

we have $\nabla^g_X\eta_i = X \lrcorner F_i$ and get

$$\begin{aligned}\nabla^g_{e_j}\Psi = {} & (e_j\lrcorner F_1)\wedge F_1 + (e_j\lrcorner F_2)\wedge F_2 + (e_j\lrcorner F_3)\wedge F_3 \\ & + 4(e_j\lrcorner F_1)\wedge\eta_2\wedge\eta_3 + 4\eta_1\wedge(e_j\lrcorner F_2)\wedge\eta_3 + 4\eta_1\wedge\eta_2\wedge(e_j\lrcorner F_3).\end{aligned}$$

This gives us

$$\begin{aligned}\nabla^g_{e_1}\Psi &= -\eta_{346} + \eta_{357} + \eta_{247} + \eta_{256}, & \nabla^g_{e_2}\Psi &= \eta_{345} + \eta_{367} - \eta_{147} - \eta_{156}, \\ \nabla^g_{e_3}\Psi &= -\eta_{245} - \eta_{267} + \eta_{146} - \eta_{157}, & \nabla^g_{e_4}\Psi &= 3\,(-\eta_{235} + \eta_{567} + \eta_{136} - \eta_{127}), \\ \nabla^g_{e_5}\Psi &= 3\,(\eta_{234} - \eta_{467} - \eta_{137} - \eta_{126}), & \nabla^g_{e_6}\Psi &= 3\,(-\eta_{237} + \eta_{457} - \eta_{134} + \eta_{125}), \\ \nabla^g_{e_7}\Psi &= 3\,(\eta_{236} - \eta_{456} + \eta_{135} + \eta_{124}).\end{aligned}$$

Using an appropriate computer algebra system we easily calculate

$$g(\nabla^g\Psi, *\Psi) = 180, \quad g(*\Psi, *\Psi) = 168,$$

thus the first condition of Lemma 4.11 is satisfied if $a = \frac{15}{14}$. Using the formulation given in Lemma 4.17 of the second condition one easily checks that the this condition is satisfied for any a. □

We expect that for all other values of the cone constant a, the structure is of generic class $\mathcal{U}_1 \oplus U_2$, but the system of equations that one obtains is extremely involved.

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